PLANE TRIGONOMETRY

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PREFACE

This book is intended to meet the needs of students both in colleges and in technical schools and presents the subject of trigonometry practically as it has been given for a number of years at the Carnegie Institute of Technology. In content, it is not intended to depart widely from the generally accepted material for such a course but includes, in addition, many devices and arrangements which the authors and their colleagues have been giving to their students by the lecture method. The desire to have these in written form led to the use of a mimeographed text which, after actual classroom use and subsequent revisions, has finally developed into the present book.

Our chief aim has been to produce a text which will require of the student a minimum of memorization of formulas but a maximum of development of principles. To accomplish this, the student must not only apply the principles and devices used to a variety of problems, but must also develop the theory involved, either by the method of the text or by an entirely different method. This throws the responsibility on the student of teaching himself and discourages the usual substitution-in-the-formula method.

An attempt has been made to anticipate the difficulties of the student. By means of an abundance of well-selected illustrative examples, over one hundred in number, the subject is developed by easy gradations, keeping in mind, however, that our students of trigonometry should be of comparative maturity. These examples are intended to bridge the gap between the theory and the exercises, and serve to show the student the advantage of a well-ordered arrangement.

The book abounds in well-graded problems, of which there

are nearly fourteen hundred. Drill exercises are placed at the end of nearly every article and a set of general exercises at the end of each chapter. The former have been inserted for the immediate illustration of the principles developed, while the latter are designed to enable the student to test his knowledge of the fundamentals of the subject and to challenge his ability to solve problems of greater difficulty. Thus the instructor is afforded a wide latitude in the choice of problems, the number being sufficient to allow a different selection each year for several years. Answers are given to the odd-numbered problems only so that the student may have a check on some of his solutions but is put on his own resources in others. The arrangement of the material is so flexible and the number of problems so numerous that the book is adaptable to courses of various lengths and content, as well as to different methods of teaching.

The following additional features may be noted:

(1) Angles of any magnitude are considered at the outset and the trigonometric functions of such angles are defined at once. Practice is provided in the use of angles other than acute angles.

(2) Radian measure is introduced early and used frequently throughout the text.

(3) The triangle and other problems are adapted to the use of five-place tables but can be solved by four or three-place tables. A chapter at the end of the book is devoted to the theory and use of logarithms.

(4) Certain of the proofs of fundamental theorems are shorter than in many texts, notably the novel but simple way of developing the addition formula.

(5) In inverse functions both notations have been used but emphasis has been laid on the arc-function notation.

(6) A chapter on the graphical representation of trigonometric functions and the approximate solution of equations involving such functions has been added.

The authors gratefully acknowledge their indebtedness to their colleagues of the Carnegie Institute of Technology for criticisms and suggestions. To Professor O. T. Geckeler, Head of the Department of Mathematics, they are under especial obligation for reading the first draft of the manuscript and for giving many very valuable suggestions. He has very generously placed at our disposal the experiences of a long period devoted intensively to the teaching of mathematics.

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PLANE TRIGONOMETRY

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CHAPTER I

TRIGONOMETRIC FUNCTIONS OF ANY ANGLE

1. Introduction. Although the greatest mathematical contributions of the ancient Greek mathematicians were in geometry, two of their astronomers, Hipparchus (about 140 B.C.) and Ptolemy (about 150 A.D.), created the science of trigonometry as a tool for their astronomical calculations. As its derivation from the two Greek words, *τρίγωνον* (trigonon) meaning triangle, and *μετρία* (metria) meaning measurement, would seem to indicate, trigonometry was originally concerned with the measurement of plane and spherical triangles, that is, having given the numerical values of three of the parts of such triangles, one of which is a side, to determine the numerical values of the other three parts. At the present time, however, the science of trigonometry has a much wider scope and although it still treats of the solution of triangles, there are many other portions of the subject which are frequently used in other branches of mathematics as well as in physics, mechanics, and engineering.

In this chapter certain preliminary topics such as directed line segments, rectangular coördinates, and angular measurement are first discussed and then immediately used in the definitions of six important ratios, called the trigonometric functions. Then follows the development of a few of the immediate properties of, and the fundamental relations between, these functions.

2. Directed line segments. It is frequently necessary in trigonometry to treat of segments of a straight line which have both magnitude and direction. Such segments are called **directed line segments.** Thus, if the direction from A to B (Fig. 1) is considered positive, then the opposite direction from B to A is negative. Expressed in symbols,

FIG. 1

$$AB = -BA \quad \text{and} \quad BA = -AB.$$

On a straight line $X'X$ let a fixed point O, called the **origin,** be taken from which to measure distances. Then with an arbitrarily chosen length as a unit, construct a numbered scale as shown in Fig. 2. If P is any point of the line, the

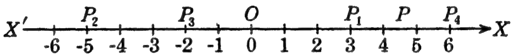


FIG. 2

symbol OP or x is used to denote the number which represents the point P. Accordingly, if P lies to the right or left of O, x is respectively a positive or negative number. To represent different points of the line, P and x with different subscripts are employed.

Thus, in Fig. 2,

$$OP_1 = x_1 = 3, \quad OP_2 = x_2 = -5, \quad OP_3 = x_3 = -2, \quad OP_4 = x_4 = 6,$$

$$\begin{aligned} P_1P_3 = P_1O + OP_3 &= -OP_1 + OP_3 \\ &= -x_1 + x_3 = -(3) + (-2) = -5, \\ P_2P_3 = P_2O - P_3O &= -OP_2 + OP_3 \\ &= -x_2 + x_3 = -(-5) + (-2) = 3. \end{aligned}$$

EXERCISE

1. In Fig. 2, find: (a) P_1P_2; (b) P_3P_2; (c) P_2P_4; (d) P_3P_1; (e) P_1P_4; (f) P_4P_3.

3. Rectangular coördinates. Let $X'X$ and $Y'Y$ be two fixed directed lines intersecting at right angles at the point O. In addition, let the positive direction be arbitrarily chosen as towards the right when parallel to $X'X$ and upwards when parallel to $Y'Y$. On each of these lines construct a numbered scale with an arbitrary length as a unit and O as the zero point of each. Then from any point P in the plane, drop perpendiculars upon $X'X$ and $Y'Y$, meeting them in M and N respectively. If x is the measure of the segment OM or NP and y of MP or ON, then the numbers x and y, called the **abscissa** and **ordinate** respectively, locate P and are called its **rectangular coördinates.** The point P whose abscissa is x and ordinate y is denoted by (x, y) or $P(x, y)$. It is to be emphasized that the abscissa and ordinate of a point, as defined, are always measured *from* $Y'Y$ and $X'X$ respectively *to* the point.

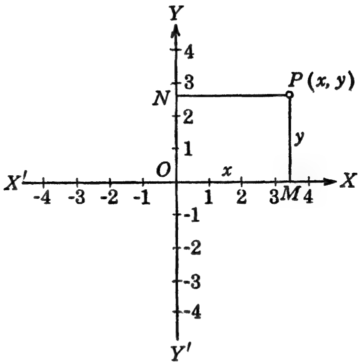


FIG. 3

The directed lines $X'X$ and $Y'Y$ are called the **axes of coördinates** and their point of intersection the **origin.** The two axes divide the whole plane into four parts, called **quadrants.** These are numbered as in Fig. 4, in which the proper signs of the coördinates are also indicated. When the

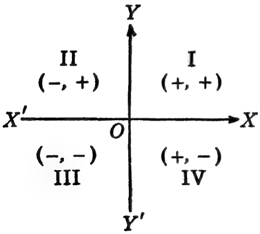


FIG. 4

coördinates of a point are known quantities the proper sign is used with the number of units. For example, a point three units to the left of $Y'Y$ and two units above $X'X$ is represented by $(-3, 2)$. If the coördinates are represented by variables, the sign does not appear. Thus a point P_3, x_3 units to the left of $Y'Y$ and y_3 units below $X'X$ would be represented by $P_3(x_3, y_3)$. In this case x_3 and y_3 represent negative numbers and if replaced by numbers, must be replaced by negative numbers.

The proper way to write the coördinates is shown for several points in Figs. 5a and 5b.

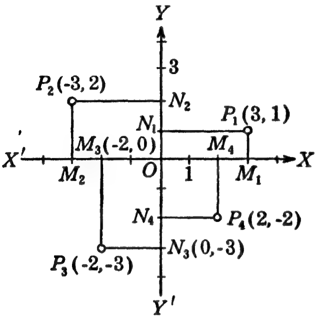


FIG. 5a

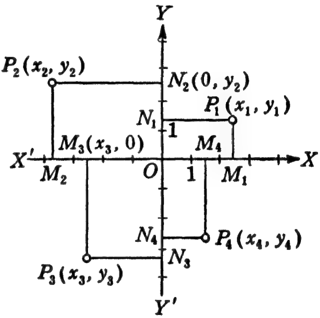


FIG. 5b

EXERCISES

Plot accurately the following points:

1. $(3, 4)$; $(-2, -4)$; $(-1, 6)$; $(0, -5)$.

2. $(3, -2)$; $(-4, 0)$; $(-5, 1)$; $(-2, -2)$.

Locate approximately the following points:

3. $(\sqrt{2}, 4)$; $(-\sqrt{3}, \sqrt{2})$; $(0, -2.7)$; $(-1\frac{1}{2}, -3\frac{3}{4})$.

4. $(0.9, -3.6)$; $(-\sqrt{5}, -2\sqrt{2})$; $(-2\frac{1}{3}, 4)$; $(\sqrt{6}, 0)$.

5. Locate $P(x, y)$ in each of the four quadrants, each on a separate figure. Is $OM = x$ and $MP = y$ wherever P may be located? If so, why? What are the values of MO and PM?

6. In Fig. 5a, find: (a) M_1M_4; (b) N_2N_3; (c) M_1M_2; (d) N_3N_4; (e) M_1M_3; (f) N_2N_4.

7. In Fig. 5b, find: (a) N_4N_1; (b) M_3M_2; (c) N_2N_1; (d) M_4M_3; (e) N_3N_1; (f) M_4M_2.

8. The hour hand of a clock is 2 in. long. Find the coördinates of its outer end referred to the horizontal and vertical diameters of the clock's face at: (a) 3 A.M.; (b) 8 P.M.; (c) 4:30 P.M.; (d) noon; (e) 10:30 A.M.

4. Angle defined. In geometry an angle is defined as the opening between two straight lines drawn from the same point. This definition is not sufficiently general for the purpose of trigonometry where it is necessary to be able to speak of angles of any magnitude, positive or negative. Such a conception of angles may be formed as follows:

An angle may be considered as generated by the rotation of a line about one of its points; the original position of the line being called the ***initial side*** *and the final position the* ***terminal side.***

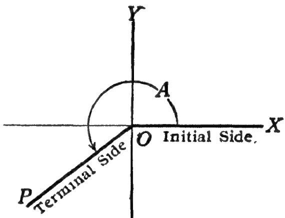


Fig. 6

Thus, in Fig. 6, suppose a line OP to start from OX, the initial line, and to rotate about O in a counter-clockwise direction to the terminal position OP. Then the angle A, indicated by the arrow, has been generated. If, in addition, the counter-clockwise direction of rotation be taken as positive, then the arrow also indicates that angle A is **positive.** Where the rotation of OP is clockwise, the angle generated is negative.

Even if angles have the same initial and terminal sides, and have been generated in the same direction, they may be different. For example, in Fig. 7, angles B and C have the same initial and terminal sides, yet $B = 90°$, and $C = 450°$.

Angle D, which equals $-180°$, is a negative angle.

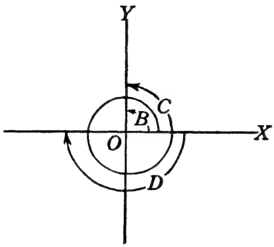


FIG. 7

An angle, positive or negative, which has the same initial side and the same terminal side as angle A is said to be **coterminal** with A. If angle A, in Fig. 6, is increased or decreased by 360°, 720°, or any multiple of 360°, the resulting angle will be coterminal with it. Hence, if n is any integer, positive or negative, then all angles coterminal with any angle A are denoted by

$$A + n \cdot 360°.$$

Thus, the angles coterminal with 120° are $120° + 1 \cdot 360°$ or 480°, $120° - 1 \cdot 360°$ or $-240°$, $120° + 2 \cdot 360°$ or 840°, $120° - 2 \cdot 360°$ or $-600°$, etc.

If, as has been done throughout this book, OX is chosen as the initial side and O the **vertex**, then an angle is said to lie in the quadrant in which its terminal side lies. For example, 125° lies in the second quadrant, 262° in the third quadrant, and $-399°$ in the fourth quadrant. Various notations are used to designate the quadrant in which an angle lies. Thus, $180° < A < 270°$, A_3, and A_{III} each denote positive third quadrant angles. Angles which are multiples of 90° do not lie in any quadrant and are called **quadrantal angles.**

To add two angles graphically, place them in the same plane with a common vertex, the initial side of the second on the terminal side of the first, each angle retaining its own direction. Then the angle from the initial side of the first to the terminal side of the second is their sum. To subtract two angles, add the negative of the subtrahend to the minuend.

To illustrate, the graphical addition of (a) A_2 and $-270°$ and (b) 180° and $-A_2$ are shown in Figs. 8a and 8b.

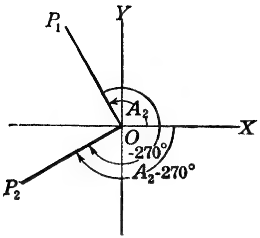


FIG. 8a

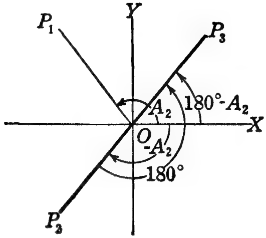


FIG. 8b

EXERCISES

With a protractor, construct the following angles and state the quadrant in which each lies:

1. 21°; 135°; −173°; 450°.
2. 166°; −18°; 540°; −122°.
3. 630°; −204°; −395°; 85°.
4. −317°; 480°; 582°; −700°.

Add the following angles graphically:

5. 120° and 45°.
6. 270° and −60°.
7. −400° and −92°.
8. −200° and 370°
9. 327° and −125°.
10. −640° and 222°.

If A_1, A_2, A_3, A_4 represent any four positive angles, each in its respective quadrant, add graphically:

11. 90° and A_1.
12. 90° and $-A_3$.
13. 180° and $-A_2$.
14. −180° and A_4.
15. −270° and $-A_2$.
16. 270° and $-A_3$.
17. −360° and A_1.
18. 360° and $-A_4$.
19. −450° and A_3.

Find three positive and three negative angles, each of which is coterminal with:

20. 63°.
21. 309°.
22. −218°.
23. −155° 30′.
24. 269° 17′.7.
25. −344°.
26. −87°.
27. 147° 45′ 33″.

5. Angular measurement. There are two systems in common use for measuring angles, namely, the **degree** or **sexagesimal system** and the **radian** or **circular measure system.**

In the sexagesimal system, a **degree** is defined as one ninetieth of a right angle. Each degree is divided into sixty equal parts called **minutes** and each minute into sixty equal parts called **seconds.** The symbols °, ′, ″ are employed to denote degrees, minutes, and seconds respectively. Thus, 47 degrees, 16 minutes, and 37 seconds is written as 47° 16′ 37″ or approximately 47° 16′.6.

In theoretical investigations, angles are not measured in degrees but in terms of a more convenient unit called a **radian.**

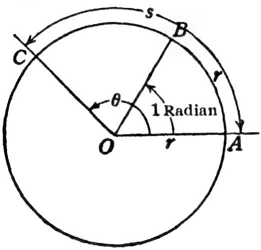


Fig. 9

A radian is defined as the angle at the center of a circle which intercepts an arc equal in length to the radius.

Thus, in Fig. 9, if the arc AB is equal to the radius OA or r, the central angle AOB is 1 radian.

From the theorem in geometry which states that in the same circle or in equal circles, the angles at the center are proportional to the intercepted arcs, it follows from Fig. 9 that

$$\frac{\text{angle } AOC}{\text{angle } AOB} = \frac{\text{arc } AC}{\text{arc } AB}. \tag{1}$$

Since angle AOB = 1 radian when arc $AB = r$, then, if the angle AOC is denoted by θ* and the arc AC by s, this proportion becomes

$$\frac{\theta}{1} = \frac{s}{r}, \tag{2}$$

or

$$s = r\theta. \qquad [1]$$

* Greek letters are frequently used to represent angles. A few of them are listed below:

Letters	*Names*	*Letters*	*Names*
α	Alpha	θ	Theta
β	Beta	ϕ	Phi
δ	Delta	ψ	Psi

That is, *the length of an arc of a circle is equal to the product of the radius and the angle at the center measured in radians.*

If the central angle is 180°, the corresponding value of s is the length of the semi-circumference of the circle or πr. Equation [1] then becomes

$$\pi r = r\theta, \tag{3}$$

or

$$\theta = \pi \text{ radians}. \tag{4}$$

$$\therefore\ \boldsymbol{\pi} \textbf{ radians} = \mathbf{180°}. \tag{5}$$

Hence,

$$1 \text{ radian} = \frac{180°}{\pi} = 57.29578°- = 57°\,17'\,44''.8+. \tag{6}$$

Conversely,

$$1° = \frac{\pi}{180} \text{ radians} = 0.0174533- \text{ radians}. \tag{7}$$

The results given in equations (6) and (7) are called conversion factors.

To facilitate the conversion of degrees to radians and of radians to degrees, special conversion tables have been appended.* Table II converts directly degrees, minutes, and seconds to radians and Table III changes radians to degrees, minutes, and seconds. These tables are to be used for conversion unless the contrary is stated.

Example 1. Express 210° in radians as a multiple of π.

$$210° = 210 \times \frac{\pi}{180} \text{ radians} = \frac{7\pi}{6} \text{ radians} = \frac{7\pi}{6}.†$$

Example 2. Convert 110° 13′ 12″ to radians correct to five decimal places by both methods.

Using the conversion factor given in (7),

$$\begin{aligned} 110°\,13'\,12'' &= 110.22 \times 0.0174533 \text{ radians} \\ &= 1.92370 \text{ radians, to five decimal places.} \end{aligned}$$

* See the Table of Contents.

† The word "radians" is usually omitted and 210° is said to be $\frac{7\pi}{6}$.

From Table II,

$$\begin{aligned}
110^\circ &= 1.9198622 \text{ radians},\\
13' &= 0.0037815 \text{ radians},\\
12'' &= 0.0000582 \text{ radians}.\\
\hline
\therefore\ 110^\circ\, 13'\, 12'' &= 1.9237019 \text{ radians}\\
&= 1.92370 \text{ radians, to five decimal places.}
\end{aligned}$$

Example 3. Express $\frac{19\pi}{6}$ in degrees.

$$\frac{19\pi}{6} = \frac{19\pi}{6} \times \frac{180^\circ}{\pi} = 570^\circ.$$

Example 4. Express 2.77 radians in degrees, minutes, and seconds.

Using the conversion factor given in (6),

$$\begin{aligned}
2.77 \text{ radians} &= 2.77 \times 57.29578^\circ = 158.70931^\circ\\
&= 158^\circ\, 42'\, 33''.5.
\end{aligned}$$

From Table III,

$$\begin{aligned}
2 \text{ radians} &= 114^\circ\, 35'\, 29''.6,\\
0.7 \text{ radians} &= 40^\circ\, 6'\, 25''.4,\\
0.07 \text{ radians} &= 4^\circ\, 0'\, 38''.5.\\
\hline
\therefore\ 2.77 \text{ radians} &= 158^\circ\, 42'\, 33''.5.
\end{aligned}$$

Example 5. Find the length, correct to five significant figures, of the arc of a circle of radius 10 in. which subtends an angle of $110^\circ\, 13'.2$ at the center.

Substituting in formula [1], $r = 10$ and $\theta = 1.92370$, which is $110^\circ\, 13'.2$ converted to radians as found in Example 2,

$$s = 10 \times 1.92370 \text{ in.} = 19.237 \text{ in.}$$

Example 6. In a circle whose radius is 10.5 cm., the length of an intercepted arc is 29.085 cm. Find the central angle (a) in radians; (b) in degrees, minutes, and seconds.

Substituting in formula [1], $r = 10.5$ and $s = 29.085$,

$$\begin{aligned}
\theta &= \frac{29.085}{10.5} = 2.77 \text{ radians}\\
&= 158^\circ\, 42'\, 33''.5, \text{ as found in Example 4.}
\end{aligned}$$

EXERCISES

In what quadrant does each of the following angles lie:

1. $\frac{3\pi}{4}$; 6; $-\frac{11\pi}{6}$; $3\frac{3}{4}$?

2. $1-\pi$; $\frac{\pi}{8}$; 1.58; $-\frac{7\pi}{3}$?

3. $\frac{13\pi}{16}$; -4.2; $\frac{4\pi+3}{5}$; $\frac{6}{\pi}$?

Express each of the following angles in radians as a multiple of π:

4. $135°$. **6.** $270°$. **8.** $-22\frac{1}{2}°$. **10.** $37° 30'$. **12.** $100° 20' 24''$.
5. $-330°$. **7.** $450°$. **9.** $-585°$. **11.** $930°$. **13.** $-310° 36'.6$.

Convert each of the following angles to radian measure, correct to five decimal places, by both methods:

14. $37° 15' 36''$. **17.** $306° 6'.6$. **20.** $289° 32'.7$.
15. $166° 29'.4$. **18.** $-118° 51' 45''$. **21.** $2' 23''$.
16. $-189° 55'.8$. **19.** $377° 15' 18''$. **22.** $-1° 16'.2$.

Convert each of the following angles to degrees, minutes, and seconds by either method:

23. $\frac{7\pi}{6}$. **26.** $-\frac{13\pi}{3}$. **29.** $\frac{2+3\pi}{5}$. **32.** $2\frac{3}{5}$.
24. 1.42. **27.** 6.293. **30.** -0.1437. **33.** -1.5708.
25. $-4\frac{1}{2}$. **28.** $\frac{3-\pi}{8}$. **31.** 0.0016. **34.** $\frac{17\pi}{6}$.

Find three positive and three negative angles in radians, each of which is coterminal with:

35. $\frac{\pi}{8}$. **37.** $\frac{7\pi}{3}$. **39.** $-\frac{\pi}{4}$. **41.** $\frac{3\pi}{5}$.
36. $-\frac{4\pi}{7}$. **38.** $\frac{5\pi}{6}$. **40.** $\frac{2\pi}{3}$. **42.** $-\frac{3\pi}{2}$.

Find three positive and three negative angles in radians as a decimal, correct to as many decimal places as the angle given, each of which is coterminal with:

43. 3.06 **44.** 0.2507. **45.** -1.143. **46.** $2\frac{3}{5}$.

In the following exercises, all angles in radian measure are to be expressed correct to five decimal places and all lengths to five significant figures.

47. In radian measure two angles of a triangle are $\frac{1}{4}$ and $\frac{1}{5}$. Find the third angle in degrees, minutes, and seconds.

48. Find the length of the arc subtending an angle of $137° 14' 33''$ at the center of a circle whose radius is 11.2 in.

49. An angle of $344° 33'.9$ at the center of a circle intercepts an arc of 37.142 cm. Find the radius.

50. Find the number of degrees, minutes, and seconds in an angle at the center of a circle of diameter 15 in. if its intercepted arc is 9.15 in.

51. Find the length of the radius of a circle at whose center an angle of $113° 22' 12''$ is subtended by an arc of 3.0976 ft.

52. The radius of a circle is 2.223 cm. Find the length of an arc which subtends a central angle of $207° 57'.7$.

53. Find the number of degrees, minutes, and seconds in an inscribed angle subtended by an arc of 15.6 cm. in a circle whose diameter is 6.5 cm.

54. The diameter of a graduated circle is 10 in., and the graduations on the circumference are 10 minutes apart. Find the distance between successive graduations.

55. Find the radius of a graduated circle if the distance between graduations 10 minutes apart is to be $\frac{1}{8}$ in.

56. The driving wheel of a locomotive is 8 ft. in diameter. If it makes 160 r.p.m., find the speed of the train in mi. per hr.

6. Definitions of the trigonometric functions. Of the greatest practical importance in all branches of pure and applied mathematics are the definitions upon which the entire subject of trigonometry is based.

To this end, consider an angle θ in each of the four quadrants as shown in Fig. 10. In each of these cases take any point $P(x, y)$ on the terminal side of this angle. Draw the perpendicular MP from P upon $X'X$. This gives three directed line segments OP, OM, and MP. The directed line segment OP, denoted by r, is called the **distance** of P

and is always taken as *positive,* being measured from the origin outward. The coördinates x or OM and y or MP are positive or negative according to the conventions of Art. 3. But it should also be remembered that the co-

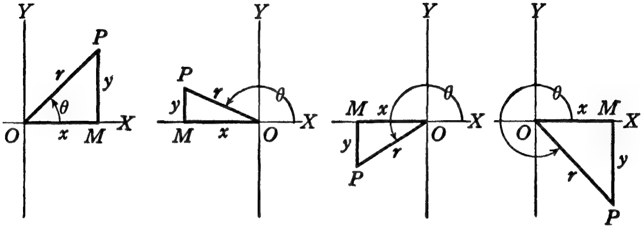


FIG. 10

ordinates of a point are always measured *from* the axes *to* the point, that is, the abscissa is always read from O to M and the ordinate from M to P; never MO or PM.

From the three quantities x, y, r, six different ratios, called the **trigonometric functions** of the angle θ, can be formed. Irrespective of the quadrant in which θ lies, they are defined as follows:

$$\textbf{sine}\ \theta\ (\text{or}\ \sin\theta) = \frac{\text{ordinate}}{\text{distance}} = \frac{MP}{OP} = \frac{y}{r},$$

$$\textbf{cosine}\ \theta\ (\text{or}\ \cos\theta) = \frac{\text{abscissa}}{\text{distance}} = \frac{OM}{OP} = \frac{x}{r},$$

$$\textbf{tangent}\ \theta\ (\text{or}\ \tan\theta) = \frac{\text{ordinate}}{\text{abscissa}} = \frac{MP}{OM} = \frac{y}{x},$$

$$\textbf{cotangent}\ \theta\ (\text{or}\ \text{ctn}\,\theta) = \frac{\text{abscissa}}{\text{ordinate}} = \frac{OM}{MP} = \frac{x}{y},$$

$$\textbf{secant}\ \theta\ (\text{or}\ \sec\theta) = \frac{\text{distance}}{\text{abscissa}} = \frac{OP}{OM} = \frac{r}{x},$$

$$\textbf{cosecant}\ \theta\ (\text{or}\ \csc\theta) = \frac{\text{distance}}{\text{ordinate}} = \frac{OP}{MP} = \frac{r}{y}.$$

In addition to these functions, the following are sometimes used:

$$\begin{aligned} \textbf{versed sine } \theta \text{ (or } \textbf{vers } \theta) &= 1 - \cos\theta, \\ \textbf{coversed sine } \theta \text{ (or } \textbf{covers } \theta) &= 1 - \sin\theta, \\ \textbf{haversine } \theta \text{ (or } \textbf{havers } \theta) &= \frac{1-\cos\theta}{2} = \frac{1}{2}\text{vers }\theta, \\ \textbf{external secant } \theta \text{ (or } \textbf{exsec } \theta) &= \sec\theta - 1. \end{aligned}$$

In order to show that the ratios are independent of the position of P, let P_1 be any other point on OP, and M_1P_1 the perpendicular from P_1 upon $X'X$. Then in any quadrant, say the third as shown in Fig. 11, the triangles OMP and OM_1P_1 are similar and the corresponding ratios $\frac{M_1P_1}{OP_1}, \frac{OM_1}{OP_1}, \frac{M_1P_1}{OM_1}$, etc. and $\frac{MP}{OP}, \frac{OM}{OP}, \frac{MP}{OM}$, etc. are equal. Hence the values of these ratios do not depend upon the position of the point P upon OP, but only upon the size of the angle θ. Since to every value of θ there corresponds one value for each of the trigonometric ratios, the ratios are called trigonometric functions of θ.

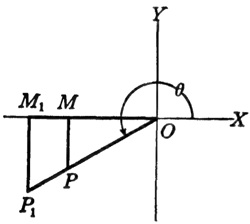


FIG. 11

Since the distance r is in all cases considered positive, the signs of the trigonometric functions of an angle in any quadrant will then depend only upon the signs of x and y. For example, consider an angle θ in the third quadrant. Then,

$$\sin\theta = \frac{y}{r} = \frac{-}{+} = -,$$

$$\cos\theta = \frac{x}{r} = \frac{-}{+} = -,$$

$$\tan\theta = \frac{y}{x} = \frac{-}{-} = +, \text{ etc.}$$

The complete results for the signs of the trigonometric functions are given in the accompanying table:

Quad-rant	sin θ	cos θ	tan θ	ctn θ	sec θ	csc θ
	$\frac{+}{+} = +$	$\frac{+}{+} = +$	$\frac{+}{+} = +$	$\frac{+}{+} = +$	$\frac{+}{+} = +$	$\frac{+}{+} = +$
II	$\frac{+}{+} = +$	$\frac{-}{+} = -$	$\frac{+}{-} = -$	$\frac{-}{+} = -$	$\frac{+}{-} = -$	$\frac{+}{+} = +$
III	$\frac{-}{+} = -$	$\frac{-}{+} = -$	$\frac{-}{-} = +$	$\frac{-}{-} = +$	$\frac{+}{-} = -$	$\frac{+}{-} = -$
IV	$\frac{-}{+} = -$	$\frac{+}{+} = +$	$\frac{-}{+} = -$	$\frac{+}{-} = -$	$\frac{+}{+} = +$	$\frac{+}{-} = -$

EXERCISES

1. Define the trigonometric functions in three different ways for the angles: (a) α_2; (b) β_4; (c) $0° < \psi < 90°$; (d) $3\pi < \phi < \frac{7\pi}{2}$; (e) $180° + A_3$; (f) $90° - A_4$; (g) $3\pi + \theta_2$.

Give the algebraic signs of the trigonometric functions of the following angles:

2. $205°$. **5.** $-122° 3' 45''$. **8.** 6.33. **11.** -4.
3. $113° 15'.6$. **6.** $2 - \pi$. **9.** $-529°$. **12.** $767°$.
4. $\frac{11\pi}{6}$. **7.** $-\frac{5\pi}{4}$. **10.** $-\frac{\pi}{3}$. **13.** $\frac{3\pi}{5}$.

In which quadrants may an angle lie, if:

14. its sine is positive? **17.** its cosine is positive?
15. its tangent is negative? **18.** its cotangent is positive?
16. its cosecant is negative? **19.** its secant is negative?

20. its sine and cotangent have the same sign?
21. its secant and tangent have opposite signs?

In what quadrant must an angle lie, if:

22. its cosine is positive and its sine is negative?
23. its secant and cotangent are both negative?
24. all the functions are positive?
25. its cosecant is negative and its tangent is positive?

7. Functions of special angles. There are a few angles which occur very frequently in problems solvable by trigonometric methods for which it is possible to find the exact values of the trigonometric functions.

For the angles 45°, 135°, 225°, or 315°, the triangle OMP (Fig. 12) is an isosceles right-angled triangle. So if OM

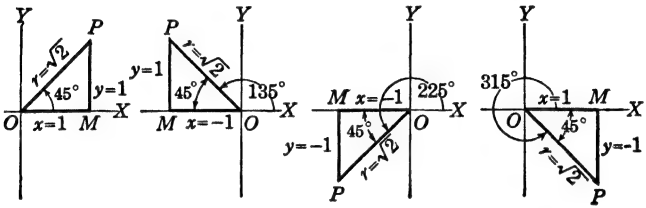


FIG. 12

and MP, the equal sides of the triangle, are each arbitrarily made of length* 1, the distance OP would be equal to $\sqrt{2}$. Therefore, from Fig. 12,

$$\sin 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}, \quad \cos 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}, \quad \tan 45° = 1,$$

$$\sin 135° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}, \quad \cos 135° = \frac{-1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}, \quad \tan 135° = -1,$$

$$\sin 225° = \frac{-1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}, \quad \cos 225° = \frac{-1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}, \quad \tan 225° = 1,$$

$$\sin 315° = \frac{-1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}, \quad \cos 315° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}, \quad \tan 315° = -1,$$

etc.

In order to find the functions of 60°, 120°, 240°, or 300°, construct an equilateral triangle ORP having each side of the triangle of length 2 as shown in Fig. 13. The perpendicular MP bisects the angle OPR and the side OR. Hence

* That is the numerical value only.

OM is of length 1 and MP of length $\sqrt{3}$. Therefore, from Fig. 13,

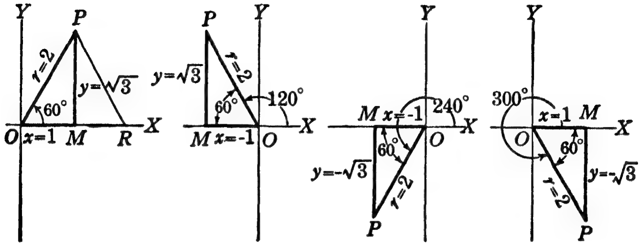


FIG. 13

$$\sin 60° = \frac{\sqrt{3}}{2}, \qquad \cos 60° = \tfrac{1}{2}, \qquad \tan 60° = \sqrt{3},$$

$$\sin 120° = \frac{\sqrt{3}}{2}, \qquad \cos 120° = -\tfrac{1}{2}, \qquad \tan 120° = -\sqrt{3},$$

$$\sin 240° = -\frac{\sqrt{3}}{2}, \qquad \cos 240° = -\tfrac{1}{2}, \qquad \tan 240° = \sqrt{3},$$

$$\sin 300° = -\frac{\sqrt{3}}{2}, \qquad \cos 300° = \tfrac{1}{2}, \qquad \tan 300° = -\sqrt{3}, \text{ etc.}$$

The functions of 30°, 150°, 210°, or 330° are easily found by a similar construction as shown in Fig. 14. Therefore,

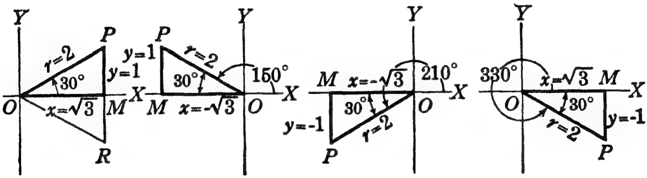


FIG. 14

$$\sin 30° = \tfrac{1}{2}, \qquad \cos 30° = \frac{\sqrt{3}}{2}, \qquad \tan 30° = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3},$$

$$\sin 150° = \tfrac{1}{2}, \qquad \cos 150° = -\frac{\sqrt{3}}{2}, \tan 150° = \frac{1}{-\sqrt{3}} = -\frac{\sqrt{3}}{3},$$

$\sin 210° = -\frac{1}{2}$, $\cos 210° = -\dfrac{\sqrt{3}}{2}$, $\tan 210° = \dfrac{-1}{-\sqrt{3}} = \dfrac{\sqrt{3}}{3}$,

$\sin 330° = -\frac{1}{2}$, $\cos 330° = \dfrac{\sqrt{3}}{2}$, $\tan 330° = \dfrac{-1}{\sqrt{3}} = -\dfrac{\sqrt{3}}{3}$,

etc.

These results are extremely important. They can be easily obtained by either actually drawing the proper figure or by forming a mental picture of that figure.

As discussed in Art. 4, all angles coterminal with any angle A are denoted by $A + n \cdot 360°$, where n is any integer, positive or negative. Since such angles have the same initial side OX and the same terminal side as angle A, any trigonometric function of $(A + n \cdot 360°)$ is equal to the same trigonometric function of A. For example, $\sin 750° = \sin 390° = \sin 30° = \frac{1}{2}$; $\tan(-585°) = \tan(-225°) = \tan 135° = -1$.

EXERCISES

In each of the following exercises, the proper figure or figures should accompany each problem.

Find the values of the trigonometric functions of the following angles:

1. $-45°$. **3.** $-150°$. **5.** $850°$. **7.** $495°$. **9.** $-480°$.

2. $\dfrac{\pi}{3}$. **4.** $\dfrac{\pi}{6}$. **6.** $\dfrac{9\pi}{4}$. **8.** $-\dfrac{13\pi}{3}$. **10.** $\dfrac{11\pi}{6}$.

Find the positive angles less than 360° for which:

11. $\operatorname{ctn}\theta = -1$. **15.** $\operatorname{ctn}\theta = -\sqrt{3}$. **19.** $\operatorname{exsec}\theta = -3$.

12. $\sin\theta = \frac{1}{2}$. **16.** $\cos\theta = -\dfrac{\sqrt{3}}{2}$. **20.** $\operatorname{vers}\theta = \frac{1}{2}$.

13. $\sec\theta = 2$. **17.** $\csc\theta = -\dfrac{2\sqrt{3}}{3}$. **21.** $\operatorname{covers}\theta = \frac{3}{2}$.

14. $\tan\theta = \dfrac{\sqrt{3}}{3}$. **18.** $\sin\theta = -\dfrac{\sqrt{2}}{2}$. **22.** $\operatorname{havers}\theta = \frac{3}{4}$.

Evaluate:

23. $\dfrac{\sin(-120°) + \cos 300° + \tan 60°}{2 \operatorname{ctn}\frac{\pi}{4} + \tan 315°}$.

24. $\dfrac{\sin 120° \operatorname{ctn} 330°}{\tan 225° - \cos\frac{\pi}{3}}$.

25. $\cos(-120°) + \tan 150° - \sin\frac{7\pi}{6} + \operatorname{ctn} 315°$.

26. $\dfrac{\cos\left(-\frac{4\pi}{3}\right) - \sin\frac{2\pi}{3} + \tan 300°}{\operatorname{ctn}\left(-\frac{5\pi}{6}\right)}$.

27. $\dfrac{\sin 930° \cos(-600°)}{\tan\frac{11\pi}{4}}$.

28. $\left[\sec^2(-510°) - \operatorname{ctn}\frac{\pi}{4}\right] \div \left[\csc 960° \operatorname{vers} 120°\right]$.

8. Given a function of an angle, to find the other functions. Consider the following problems in which a function of an angle is given.

Example 1. Given $\cos\theta = -\frac{12}{13}$, construct the angle θ and find the other functions.

By definition $\cos\theta = \dfrac{x}{r}$ so that $\dfrac{x}{r} = \dfrac{-12}{13}$. As r is always positive, the minus sign must be taken with the abscissa. When $r = 13$, $x = -12$, and $y = \pm\sqrt{169 - 144} = \pm 5$. Therefore θ must lie in the second or third quadrants.

Construct two figures, Figs. 15a and 15b, in each of which a line is drawn parallel to and 12 units to the left of $Y'Y$. Then with O as a center and a radius of 13 units, describe an arc of a circle intersecting the line in P_2 in one figure and P_3 in the other. Join O to P_2 and P_3. Two triangles OM_2P_2 and OM_3P_3 are thus formed, in each of which the ratio $\dfrac{x}{r} = -\dfrac{12}{13}$. Call the two positive angles

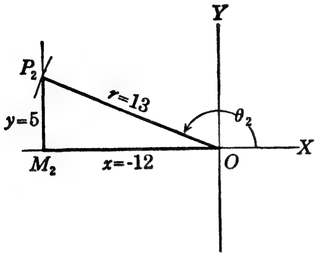


FIG. 15a

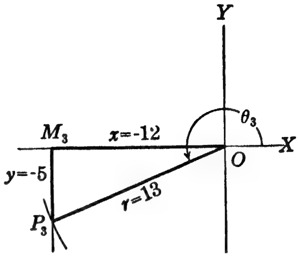


FIG. 15b

θ_2 and θ_3. Hence, the functions are:

Second quadrant	Third quadrant
$\sin\theta_2 = \frac{5}{13}$,	$\sin\theta_3 = -\frac{5}{13}$,
$\cos\theta_2 = -\frac{12}{13}$,	$\cos\theta_3 = -\frac{12}{13}$,
$\tan\theta_2 = -\frac{5}{12}$,	$\tan\theta_3 = \frac{5}{12}$,
$\operatorname{ctn}\theta_2 = -\frac{12}{5}$,	$\operatorname{ctn}\theta_3 = \frac{12}{5}$,
$\sec\theta_2 = -\frac{13}{12}$,	$\sec\theta_3 = -\frac{13}{12}$,
$\csc\theta_2 = \frac{13}{5}$,	$\csc\theta_3 = -\frac{13}{5}$.

Example 2. Given $\operatorname{ctn}\phi = -\frac{24}{7}$, construct the angle ϕ and find the other functions.

Since $\operatorname{ctn}\phi = \frac{x}{y}$, then $\frac{x}{y} = -\frac{24}{7} = \frac{-24}{7} = \frac{24}{-7}\cdot$ Hence when $x = -24$, $y = 7$ and when $x = 24$, $y = -7$. Therefore ϕ must lie in the second or fourth quadrants. In either case

$$r = \sqrt{(\mp 24)^2 + (\pm 7)^2} = \sqrt{625} = 25.$$

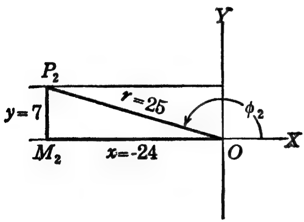


FIG. 16a

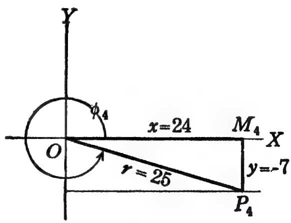


FIG. 16b

Construct the two figures as shown in Figs. 16a and 16b. Therefore, the functions are:

Second quadrant	Fourth quadrant
$\sin \phi_2 = \frac{7}{25}$,	$\sin \phi_4 = -\frac{7}{25}$,
$\cos \phi_2 = -\frac{24}{25}$,	$\cos \phi_4 = \frac{24}{25}$,
$\tan \phi_2 = -\frac{7}{24}$,	$\tan \phi_4 = -\frac{7}{24}$,
$\operatorname{ctn} \phi_2 = -\frac{24}{7}$,	$\operatorname{ctn} \phi_4 = -24$,
$\sec \phi_2 = -\frac{25}{24}$,	$\sec \phi_4 = \frac{25}{24}$,
$\csc \phi_2 = \frac{25}{7}$,	$\csc \phi_4 = -\frac{25}{7}$.

EXERCISES

Construct the angle θ and find the other functions when given that:

1. $\csc \theta = -\frac{13}{5}$.
2. $\cos \theta = \frac{8}{17}$.
3. $\tan \theta = -\frac{2}{3}$.
4. $\sec \theta = -\frac{25}{7}$.
5. $\sin \theta = \frac{4}{7}$.
6. $\operatorname{ctn} \theta = \frac{35}{12}$.
7. $\tan \theta = -\frac{60}{11}$ and $\sin \theta > 0$.
8. $\operatorname{vers} \theta = \frac{49}{29}$ and $180° < \theta < 270°$.
9. $\operatorname{covers} \theta = \frac{32}{25}$ and $\tan \theta < 0$.
10. $\sec \theta = -4$ and $\frac{\pi}{2} < \theta < \pi$.
11. $\sin \theta = -\frac{40}{41}$ and $270° < \theta < 360°$.
12. $\operatorname{ctn} \theta = -\pi$ and $\csc \theta < 0$.

13. Find the value of $\dfrac{\operatorname{ctn} \theta + \csc \theta}{\sec^2 \theta - \tan \theta}$, when $\sin \theta = \frac{3}{5}$ and $90° < \theta < 180°$.

14. Find the value of $\left[\dfrac{2 - \operatorname{ctn} \theta}{\csc \theta + \cos \theta}\right]^2$, when $\tan \theta = 2$ and $\cos \theta < 0$.

15. Find the value of $\sqrt{\operatorname{vers} \theta + \sin^2 \theta} \cdot (\csc^2 \theta - \operatorname{ctn}^2 \theta)^3$, when $\cos \theta = -\frac{1}{3}$ and $\tan \theta < 0$.

16. Find the value of $\left[\dfrac{\tan \theta + \sec \theta + 1}{\operatorname{ctn} \theta - \csc \theta - 1}\right]^{-1}$, when $\csc \theta = -\frac{13}{5}$ and $\dfrac{3\pi}{2} < \theta < 2\pi$.

9. Fundamental relations between the functions of an angle. The six trigonometric functions are connected by certain fundamental relations. It is the purpose of this article to derive these relations.

From the very definitions of these functions, the **reciprocal relations** listed below. follow directlv:

$$\csc\theta = \frac{1}{\sin\theta} \quad \text{and} \quad \sin\theta = \frac{1}{\csc\theta}, \tag{2}$$

$$\sec\theta = \frac{1}{\cos\theta} \quad \text{and} \quad \cos\theta = \frac{1}{\sec\theta}, \tag{3}$$

$$\operatorname{ctn}\theta = \frac{1}{\tan\theta} \quad \text{and} \quad \tan\theta = \frac{1}{\operatorname{ctn}\theta}. \tag{4}$$

Hence, *the sine and cosecant, the cosine and secant, the tangent and cotangent respectively of the same angle are called* ***reciprocal functions.***

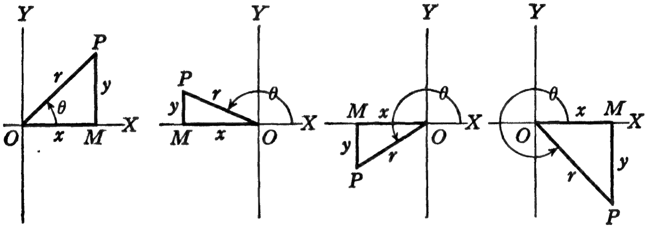


Fig. 17

Let θ be an angle in any quadrant (Fig. 17). Then, irrespective of the quadrant in which it lies,

$$x^2 + y^2 = r^2. \tag{1}$$

Dividing both sides of (1) by r^2,

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1.$$

But

$$\frac{x}{r} = \cos\theta \quad \text{and} \quad \frac{y}{r} = \sin\theta.$$

$$\therefore\ (\cos\theta)^2 + (\sin\theta)^2 = 1.$$

This is usually written in the form

$$\sin^2\theta + \cos^2\theta = 1. \tag{5}$$

Similarly, dividing both sides of (1) by x^2,

$$1 + \frac{y^2}{x^2} = \frac{r^2}{x^2}.$$

But

$$\frac{y}{x} = \tan\theta \quad \text{and} \quad \frac{r}{x} = \sec\theta.$$

$$\therefore \mathbf{1 + \tan^2\theta = \sec^2\theta.} \tag{6}$$

Dividing both sides of (1) by y^2, and changing the order of the terms,

$$1 + \frac{x^2}{y^2} = \frac{r^2}{y^2}.$$

But

$$\frac{x}{y} = \operatorname{ctn}\theta \quad \text{and} \quad \frac{r}{y} = \csc\theta.$$

$$\therefore \mathbf{1 + ctn^2\theta = csc^2\theta.} \tag{7}$$

Formulas [5], [6], and [7] are called the **Pythagorean relations.**

Again, by definition,

$$\mathbf{tan\,\theta} = \frac{y}{r} = \frac{\left(\frac{y}{r}\right)}{\left(\frac{x}{r}\right)} = \frac{\mathbf{sin\,\theta}}{\mathbf{cos\,\theta}}, \tag{8}$$

$$\mathbf{ctn\,\theta} = \frac{x}{y} = \frac{\left(\frac{x}{r}\right)}{\left(\frac{y}{r}\right)} = \frac{\mathbf{cos\,\theta}}{\mathbf{sin\,\theta}}. \tag{9}$$

Formulas [8] and [9] are called the **quotient relations.**

These eight **fundamental relations** are frequently used in trigonometry, and must be memorized.

Example. Given $\operatorname{ctn}\theta = -\frac{3}{2}$ and $270° < \theta < 360°$, find the remaining functions by means of the fundamental relations.

Since θ is given in the fourth quadrant, all the functions are negative except the cosine and secant.

By [4], $\tan\theta = \dfrac{1}{\text{ctn}\,\theta} = \dfrac{1}{-\frac{3}{2}} = -\frac{2}{3}$.

By [7], $\csc\theta = -\sqrt{1+\text{ctn}^2\theta} = -\sqrt{1+\frac{9}{4}} = -\frac{1}{2}\sqrt{13}$.

By [2], $\sin\theta = \dfrac{1}{\csc\theta} = \dfrac{1}{-\frac{1}{2}\sqrt{13}} = -\frac{2}{13}\sqrt{13}$.

By [6], $\sec\theta = \sqrt{1+\tan^2\theta} = \sqrt{1+\frac{4}{9}} = \frac{1}{3}\sqrt{13}$.

By [3], $\cos\theta = \dfrac{1}{\sec\theta} = \dfrac{1}{\frac{1}{3}\sqrt{13}} = \frac{3}{13}\sqrt{13}$.

EXERCISES

1. Derive the Pythagorean and quotient relations for the angles: (a) ϕ_3; (b) $360° < \alpha < 450°$; (c) β_4; (d) $\dfrac{5\pi}{2} < \psi < 3\pi$.

In the following exercises, find the remaining functions by means of the fundamental relations:

2. $\sin\theta = \frac{4}{5}$ and $90° < \theta < 180°$.

3. $\text{ctn}\,\theta = -\frac{12}{5}$ and $\cos\theta > 0$.

4. $\sec\theta = \frac{3}{2}$ and $0 < \theta < \dfrac{\pi}{2}$.

5. $\tan\theta = \frac{24}{7}$ and $\csc\theta < 0$.

6. $\cos\theta = -\frac{4}{7}$ and $\pi < \theta < \dfrac{3\pi}{2}$.

7. $\csc\theta = -\frac{17}{8}$ and $270° < \theta < 360°$.

10. To express each function in terms of one of them. Consider the problem of expressing each of the functions in terms of one of them. For the sake of simplicity, the given angle has been assumed to be acute. Hence all the functions are positive. Should the given angle be in any other quadrant, the proper algebraic signs of the functions could then be determined by the quadrant in which the given angle lies. The following examples illustrate the methods used.

Example 1. Express each of the functions in terms of $\sin\theta$ by means of the fundamental relations, where θ is an acute angle.

By [5], $\cos\theta = \sqrt{1-\sin^2\theta}.$

By [8], $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{\sin\theta}{\sqrt{1-\sin^2\theta}}.$

By [4], $\text{ctn}\,\theta = \dfrac{1}{\tan\theta} = \dfrac{\sqrt{1-\sin^2\theta}}{\sin\theta}.$

By [3], $\sec\theta = \dfrac{1}{\cos\theta} = \dfrac{1}{\sqrt{1-\sin^2\theta}}.$

By [2], $\csc\theta = \dfrac{1}{\sin\theta}.$

Example 2. Express each of the functions in terms of $\tan\phi$ by means of a figure, where ϕ is an acute angle.

By definition $\tan\phi = \dfrac{y}{x}$ so that $\dfrac{y}{x} = \dfrac{\tan\phi}{1}.$ Hence when $x = 1$, $y = \tan\phi$. Therefore $r = \sqrt{1+\tan^2\phi}$. The remaining functions are:

$$\sin\phi = \frac{\tan\phi}{\sqrt{1+\tan^2\phi}},$$

$$\cos\phi = \frac{1}{\sqrt{1+\tan^2\phi}},$$

$$\cot\phi = \frac{1}{\tan\phi},$$

$$\sec\phi = \sqrt{1+\tan^2\phi},$$

$$\csc\phi = \frac{\sqrt{1+\tan^2\phi}}{\tan\phi}.$$

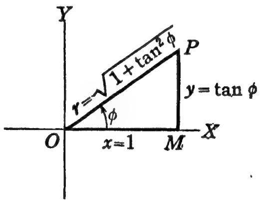


FIG. 18

EXERCISES

By means of the fundamental relations, express each of the other functions of θ, where θ is an acute angle, in terms of:

1. $\cos\theta$. **2.** $\text{ctn}\,\theta$. **3.** $\csc\theta$. **4.** $\tan\theta$.

By means of a figure, express each of the other functions of ϕ, where $0° < \phi < 90°$, in terms of:

5. $\sin\phi$. **6.** $\sec\phi$. **7.** $\tan\phi$. **8.** $\cos\phi$.

Express each of the following functions in terms of each of the other five functions, assuming θ to be acute:

9. $\operatorname{ctn}\theta$. **10.** $\cos\theta$. **11.** $\csc\theta$. **12.** $\tan\theta$. **13.** $\sec\theta$.

11. Simple trigonometric identities. As in algebra, an **identical trigonometric equation** or **trigonometric identity,** is defined as an equation which is satisfied for all possible values* of the angle or angles. Hence the eight fundamental relations are trigonometric identities since they are true for all values of the angle. By means of these relations, it is possible to change any expression containing trigonometric functions into a variety of different forms. Hence it is often necessary to be able to show that two expressions, although different in form, are nevertheless identical in value.

The truth of an identity is usually established by transforming either member, by means of known identities, to the form of the other, or by transforming both members to a common third form. However, all identities in this book are to be proved by changing the form of the left side to that of the right side.

There is no general method of procedure. Radicals should be avoided whenever possible. When some other method of attack is not suggested by the forms of the two expressions, a reduction to sines and cosines is usually effective.

Example 1. Prove that: $\dfrac{1+\tan^2\theta}{\csc^2\theta} = \tan^2\theta.$

As the second member should suggest, the proof consists in reducing the first member to a single term. This can best be effected by replacing $1+\tan^2\theta$ by $\sec^2\theta$, $\sec^2\theta$ by $\dfrac{1}{\cos^2\theta}$, $\csc^2\theta$ by $\dfrac{1}{\sin^2\theta}$,

* Any exceptional angles for which the identities become meaningless because some of the trigonometric functions involved are not defined (see Art. 27) or which make a denominator equal to zero, are excluded.

and finally $\dfrac{\sin^2\theta}{\cos^2\theta}$ by $\tan^2\theta$. These transformations are shown in the arrangement below:

$$\frac{1+\tan^2\theta}{\csc^2\theta} = \tan^2\theta.$$

$$\frac{\sec^2\theta}{\csc^2\theta} =$$

$$\frac{\dfrac{1}{\cos^2\theta}}{\dfrac{1}{\sin^2\theta}} =$$

$$\frac{\sin^2\theta}{\cos^2\theta} =$$

$$\tan^2\theta =$$

Example 2. Prove that: $\dfrac{1+\sin\theta}{\cos\theta} = \dfrac{\cos\theta}{1-\sin\theta}$.

A simple method for obtaining $(1 - \sin\theta)$, which appears as the denominator of the second member, in the denominator of the first member is to multiply both numerator and denominator of that member by $(1 - \sin\theta)$. The remaining steps in the proof are shown below:

$$\frac{1+\sin\theta}{\cos\theta} = \frac{\cos\theta}{1-\sin\theta}.$$

$$\frac{(1+\sin\theta)(1-\sin\theta)}{\cos\theta\,(1-\sin\theta)} =$$

$$\frac{1-\sin^2\theta}{\cos\theta\,(1-\sin\theta)} =$$

$$\frac{\cos^2\theta}{\cos\theta\,(1-\sin\theta)} =$$

$$\frac{\cos\theta}{1-\sin\theta} =$$

EXERCISES

Prove the following trigonometric identities:

1. $\sin\theta \operatorname{ctn}\theta = \cos\theta$.
2. $\tan\theta = \sin\theta\sec\theta$.
3. $\dfrac{\cos^2\theta}{1-\cos^2\theta} = \operatorname{ctn}^2\theta$.
4. $\tan\theta + \operatorname{ctn}\theta = \sec\theta\csc\theta$.
5. $\tan^2\theta - \sin^2\theta = \tan^2\theta\sin^2\theta$.
6. $\operatorname{ctn}\theta\csc\theta = \dfrac{1}{\sec\theta - \cos\theta}$.

7. $\sin\theta = \dfrac{\tan\theta}{\sec\theta}$.

8. $\dfrac{\sqrt{1+\tan^2\theta}}{\sqrt{1+\text{ctn}^2\,\theta}} = \tan\theta$.

9. $\dfrac{\sqrt{\sec^2\theta - 1}}{\sqrt{1-\sin^2\theta}} = \dfrac{\sec\theta}{\text{ctn}\,\theta}$.

10. $\dfrac{\text{ctn}^2\,\theta}{1+\text{ctn}^2\,\theta} = \cos^2\theta$.

11. $\dfrac{\sec\theta}{\csc^2\theta} = \sec\theta - \cos\theta$.

12. $\csc\theta = \dfrac{\sqrt{1+\tan^2\theta}}{\tan\theta}$.

13. $\dfrac{1}{\tan^2\theta+1} + \dfrac{1}{1+\text{ctn}^2\,\theta} = 1$.

14. $\dfrac{\sin\alpha}{1+\cos\alpha} = \dfrac{1-\cos\alpha}{\sin\alpha}$.

15. $\dfrac{\tan^2\alpha}{\sec\alpha+1} = \dfrac{1-\cos\alpha}{\cos\alpha}$.

16. $\sec\beta + \csc\beta = \dfrac{1+\tan\beta}{\sin\beta}$.

17. $\dfrac{\sec\beta+1}{\tan\beta} = \dfrac{\tan\beta}{\sec\beta-1}$.

18. $\dfrac{1}{\sec\theta+\tan\theta} = \sec\theta - \tan\theta$.

19. $\dfrac{1}{\csc\theta - \text{ctn}\,\theta} = \csc\theta + \text{ctn}\,\theta$.

20. $\dfrac{\text{ctn}\,\theta}{\csc\theta+1} = \dfrac{\csc\theta-1}{\text{ctn}\,\theta}$.

21. $\sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}} = \sec\theta - \tan\theta$.

22. $\dfrac{\tan\alpha+\tan\beta}{\text{ctn}\,\alpha+\text{ctn}\,\beta} = \tan\alpha\tan\beta$.

23. $\text{ctn}^2\,\alpha - \cos^2\alpha = \cos^2\alpha\,\text{ctn}^2\,\alpha$.

24. $\tan^4\beta + \tan^2\beta = \sec^4\beta - \sec^2\beta$.

25. $\cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1$.

26. $\cos^4\theta - \sin^4\theta = 1 - 2\sin^2\theta$.

27. $\csc^2\theta - \sec^2\theta = \text{ctn}^2\,\theta - \tan^2\theta$.

28. $\csc^2\theta\sec^2\theta = \csc^2\theta + \sec^2\theta$.

29. $\dfrac{1}{1-\cos\theta} + \dfrac{1}{1+\cos\theta} = 2\csc^2\theta$.

30. $\csc^2\theta - \csc\theta\,\text{ctn}\,\theta = \dfrac{1}{1+\cos\theta}$.

31. $\dfrac{1}{1+\tan^2\alpha} - \dfrac{1}{\text{ctn}^2\,\alpha+1} = 1 - 2\sin^2\alpha$.

32. $\dfrac{1}{1+\sin\theta} + \dfrac{1}{1-\sin\theta} = 2(\tan^2\theta+1)$.

33. $\dfrac{\cos\theta}{1-\tan\theta} + \dfrac{\sin\theta}{1-\text{ctn}\,\theta} = \sin\theta + \cos\theta$.

34. $\dfrac{\sin\beta}{1+\cos\beta} + \dfrac{1+\cos\beta}{\sin\beta} = 2\csc\beta$.

35. $(\text{exsec}\,\theta + \text{vers}\,\theta)(\text{exsec}\,\theta + \text{covers}\,\theta) = \dfrac{1-\sin\theta\cos\theta}{\text{ctn}^2\,\theta}$.

36. $\dfrac{1+\sin^2\theta\sec^2\theta}{1+\cos^2\theta\csc^2\theta} = \sin^2\theta\sec^2\theta.$

37. $\dfrac{(\sec\alpha+\csc\alpha)^2}{\tan\alpha+\operatorname{ctn}\alpha} = 2+\sec\alpha\csc\alpha.$

38. $\csc^4\theta\,(1-\cos^4\theta)-2\operatorname{ctn}^2\theta = 1.$

39. $2+\dfrac{\sin^4\theta+\cos^4\theta}{\sin^2\theta\cos^2\theta} = \sec^2\theta\csc^2\theta.$

40. $\dfrac{\tan^2\theta}{\operatorname{ctn}^2\theta\,(1+\tan^2\theta)^2}+1-2\,(1-\operatorname{covers}\theta)^2 = \cos^4\theta.$

GENERAL EXERCISES

1. What angle will the minute hand of a clock generate in 3 hrs. 36 min. 12 sec.?

2. In what quadrant must an angle lie, if: (a) its tangent is negative and its cosine is positive?; (b) its secant and cosecant are both negative?

3. Prove the Pythagorean and quotient relations for an angle ϕ in (a) the second quadrant; (b) the fourth quadrant.

4. Construct the angle α and find the other functions, given that $\sec\alpha = -\frac{29}{20}$.

5. Evaluate: $\dfrac{\sin 510^\circ - \sin 390^\circ + \cos\left(-\dfrac{2\pi}{3}\right)}{2\tan^3(-495^\circ)}$.

6. Add graphically: (a) 270° and $-A_4$; (b) $-\pi$ and A_2.

7. By means of a figure, express each of the other functions of β in terms of $\operatorname{ctn}\beta$, where $0^\circ < \beta < 90^\circ$.

8. Prove: $(\cos\theta+\sin\theta)^4-(\cos\theta-\sin\theta)^4 = 8\sin\theta\cos\theta.$

9. Find three positive and three negative angles each of which is coterminal with: (a) $39^\circ\,48'.7$; (b) $\dfrac{8\pi}{5}$; (c) -3.72; (d) $-244^\circ\,55'\,32''$.

10. Prove: $\dfrac{\sin^2\phi\,(1+\tan^2\phi)-\cos^2\phi\,(\operatorname{ctn}^2\phi+1)}{\tan\phi-\operatorname{ctn}\phi} = \sec\phi\csc\phi.$

11. Prove: $\operatorname{ctn}\alpha-\sec\alpha\csc\alpha\,(1-2\sin^2\alpha) = \tan\alpha.$

12. The radius of a circle is 10.47 in. Find the length of the arc which subtends a central angle of $169^\circ\,43'.8$.

13. Show that $\sin\theta < \theta$ if θ is a positive acute angle expressed in radians.

14. Express $\dfrac{\sec^2\phi - \sin^4\phi\sec^2\phi\,(1+\text{ctn}^2\phi)}{\sin^2\phi\tan^2\phi}$ in terms of $\tan\phi$.

15. Prove: $\dfrac{\tan\alpha - \sin\alpha}{\sin^2\alpha} = \dfrac{\sin\alpha\sec\alpha}{1+\cos\alpha}$.

16. Express $\sin\phi$ in terms of each of the other functions, assuming ϕ to be acute.

17. An inscribed angle of 73° 34′ 55″ intercepts an arc of 2.103 ft. Find the radius of the circle.

18. Find the value of $\dfrac{\sec A + \tan A + 1}{\cos^2 A - \sin A}$, when $\text{ctn}\, A = -\frac{5}{12}$ and $90° < A < 180°$.

19. Find the positive angles less than 360° for which: (a) $\tan\theta = -\sqrt{3}$; (b) $\cos\theta = \dfrac{\sqrt{3}}{2}$; (c) $\sin\theta = -\dfrac{\sqrt{2}}{2}$; (d) $\sec\theta = -\sqrt{2}$; (e) $\text{ctn}\,\theta = \dfrac{\sqrt{3}}{3}$; (f) $\csc\theta = 2$.

20. Add graphically: (a) $-\dfrac{\pi}{2}$ and A_3; (b) 360° and $-A_1$.

21. Prove: $\dfrac{\sin B + \sin C}{\cos B + \cos C} + \dfrac{\cos B - \cos C}{\sin B - \sin C} = 0.$

22. An angle of 30° at the center O of a circle subtends an arc BC of length $\dfrac{\pi}{3}$ ft. Find the length of the perpendicular dropped from B upon OC.

23. Express, in radians per second, the angular velocities of the hour, minute and second hands of a watch.

24. By means of the fundamental relations, determine the remaining functions when given $\tan\theta = -\frac{15}{8}$ and $\sec\theta < 0$.

25. Given $\cos\psi = \dfrac{e^m + e^{-m}}{2}$, show that $\tan\psi = \pm\dfrac{e^m - e^{-m}}{e^m + e^{-m}}$.

26. A railway train is rounding a curve of radius 1980 ft. at the rate of 20 mi. per hr. Through what angle does it turn in 1 min.?

27. Evaluate: $\dfrac{\sin\dfrac{7\pi}{3}\cdot\tan^2(-210°)}{\csc(-390°) + 3\,\text{ctn}\,\dfrac{13\pi}{4}}$.

28. The moon's distance from the earth is 238,840 mi. and its

diameter subtends an angle of 31′ 7″ at the earth. Find its diameter.

29. Prove: $\dfrac{1+\cos\theta}{\sec\theta-\tan\theta}-\dfrac{1-\cos\theta}{\sec\theta+\tan\theta}=2(1+\tan\theta)$.

30. By means of the fundamental relations, express each of the other functions of θ in terms of $\sec\theta$, where $270° < \theta < 360°$.

31. Construct the angle B and find the other functions, given that $\csc B = -\frac{41}{40}$ and $\sec B > 0$.

32. Prove: $(\csc\theta-\operatorname{ctn}\theta)^2=\dfrac{1-\cos\theta}{1+\cos\theta}$.

33. Find the number of degrees, minutes, and seconds in the central angle of a circle of radius 17 cm. which intercepts an arc of 18.887 cm.

34. Show that $\tan\theta > \theta$ if θ is a positive acute angle expressed in radians.

35. Express $\dfrac{\sec^2\theta\sin^2\theta-\csc^2\theta+\csc^2\theta\cos^2\theta}{\sec^2\theta\sin^2\theta-\csc^2\theta\cos^2\theta}$ in terms of $\sin\theta$.

36. By means of the fundamental relations, determine the remaining functions when $\cos\theta=\dfrac{2}{5}$ and $\dfrac{3\pi}{2}<\theta<2\pi$.

37. If the radius of the earth is considered as 3963.3 mi., find the number of ft. in an arc of 1° on a meridian.

38. Two angles of a triangle are $\dfrac{\pi}{8}$ and $\dfrac{1}{8}$. Find the third angle in degrees, minutes, and seconds.

39. Prove: $\dfrac{1-\sin\theta}{1+\sin\theta}=(\sec\theta-\tan\theta)^2$.

40. Two wheels of radii 13 in. and 17.5 in., respectively, are belted together. Through how many degrees, minutes, and seconds will the smaller wheel turn while the larger wheel turns through 1200°?

41. Prove: $\dfrac{\sin\theta-\operatorname{covers}\theta\tan^2\theta}{\operatorname{ctn}^2\theta\csc\theta}=\dfrac{\tan^2\theta}{(1+\sin\theta)\csc^2\theta}$.

42. If $x=r\sin\theta\cos\phi$, $y=r\sin\theta\sin\phi$, and $z=r\cos\theta$, prove that $x^2+y^2+z^2=r^2$.

43. Find the positive angles less than 360° for which

$$\cos\alpha=\frac{\tan(-300°)\sqrt[3]{\operatorname{ctn}\left(-\frac{5\pi}{4}\right)}}{\csc 690°}.$$

44. If $dx = \cos\theta\, dr - r\sin\theta\, d\theta$ and $dy = \sin\theta\, dr + r\cos\theta\, d\theta$, show that the equation $ds^2 = dx^2 + dy^2$ becomes $ds^2 = dr^2 + r^2\, d\theta^2$.

45. Find the positive angles less than 360° for which

$$\sin\phi = \frac{\cos\dfrac{19\pi}{6} - \tan(-480^\circ)}{\sec^2\left(-\dfrac{\pi}{4}\right) + \dfrac{1}{2}\csc 510^\circ}.$$

CHAPTER II

RIGHT TRIANGLES

12. Introduction. This chapter deals primarily with the trigonometric functions of acute angles and their application to the solution of right triangles. Although a systematic treatment of the solution of oblique triangles is deferred until Chapter V, a general method is given for solving such triangles by means of right triangles.

13. Functions of acute angles. For convenience, the general definitions of the trigonometric functions for any angle are restated for the angles of a right triangle.

In the right triangle ABC (Fig. 19), let a, b, c denote the lengths of the sides opposite the acute angles A, B, and the right angle C, respectively.

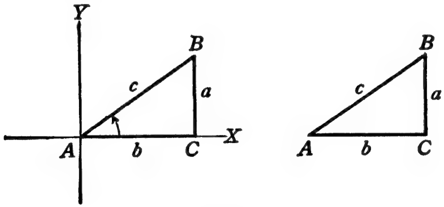


FIG. 19

Then, by definition,

$$\sin A = \frac{\text{ordinate}}{\text{distance}} = \frac{a}{c} = \frac{\text{side opposite}}{\text{hypotenuse}},$$

$$\cos A = \frac{\text{abscissa}}{\text{distance}} = \frac{b}{c} = \frac{\text{side adjacent}}{\text{hypotenuse}},$$

$$\tan A = \frac{\text{ordinate}}{\text{abscissa}} = \frac{a}{b} = \frac{\text{side opposite}}{\text{side adjacent}},$$

$$\text{ctn } A = \frac{\text{abscissa}}{\text{ordinate}} = \frac{b}{a} = \frac{\text{side adjacent}}{\text{side opposite}},$$

$$\sec A = \frac{\text{distance}}{\text{abscissa}} = \frac{c}{b} = \frac{\text{hypotenuse}}{\text{side adjacent}},$$

$$\csc A = \frac{\text{distance}}{\text{ordinate}} = \frac{c}{a} = \frac{\text{hypotenuse}}{\text{side opposite}}.$$

Similarly, for the acute angle B,

$$\sin B = \frac{b}{c} = \frac{\text{side opposite}}{\text{hypotenuse}}, \qquad \text{ctn } B = \frac{a}{b} = \frac{\text{side adjacent}}{\text{side opposite}},$$

$$\cos B = \frac{a}{c} = \frac{\text{side adjacent}}{\text{hypotenuse}}, \qquad \sec B = \frac{c}{a} = \frac{\text{hypotenuse}}{\text{side adjacent}},$$

$$\tan B = \frac{b}{a} = \frac{\text{side opposite}}{\text{side adjacent}}, \qquad \csc B = \frac{c}{b} = \frac{\text{hypotenuse}}{\text{side opposite}}.$$

By comparing these with the functions of angle A, it is evident that:

$$\sin A = \cos B, \qquad \text{ctn } A = \tan B,$$

$$\cos A = \sin B, \qquad \sec A = \csc B,$$

$$\tan A = \text{ctn } B, \qquad \csc A = \sec B.$$

Since angles A and B are complementary and the cosine, cotangent, and cosecant are called **co-functions** of the sine, tangent, and secant respectively, the above results may be stated in a theorem as follows:

A function of an acute angle is equal to the co-function of its complementary angle.

Example 1. Express (a) $\sin 63° \, 10'.6$ and (b) $\text{ctn } \frac{3\pi}{8}$ as functions of angles less than 45°.

(a) $\sin 63° \, 10'.6 = \cos (90° - 63° \, 10'.6) = \cos 26° \, 49'.4.$

(b) $\text{ctn } \frac{3\pi}{8} = \tan\left(\frac{\pi}{2} - \frac{3\pi}{8}\right) = \tan \frac{\pi}{8}.$

Example 2. Find a value of α if $\csc (5\alpha + 27°) = \sec 2\alpha.$

Changing the second member of this equation to the cosecant by means of the above theorem,

$$\begin{aligned} \csc(5\alpha + 27^\circ) &= \csc(90^\circ - 2\alpha). \\ \therefore\ 5\alpha + 27^\circ &= 90^\circ - 2\alpha, \\ 7\alpha &= 63^\circ, \\ \alpha &= 9^\circ. \end{aligned}$$

This value could also have been obtained by changing the first member to the secant.

EXERCISES

Express each of the following as functions of angles less than 45°:

1. $\sin 67^\circ\, 12'.4$.
2. $\cos \dfrac{2\pi}{7}$.
3. $\tan 83^\circ\, 36'\, 43''$.
4. $\text{ctn}\, 72^\circ\, 52'.5$.
5. $\csc 1.29$.
6. $\sec \dfrac{5\pi}{16}$.

Find a value of θ, *if:*

7. $\cos 2\theta = \sin(\theta - 60^\circ)$.
8. $\sec\left(\dfrac{4\pi}{7} - 3\theta\right) = \csc\theta$.
9. $\tan(3\theta + 10^\circ) = \text{ctn}\left(\dfrac{\theta}{2} - 60^\circ\right)$.

Show that in any triangle ABC:

10. $\cos\dfrac{A}{2} = \sin\left(\dfrac{B + C}{2}\right)$.
11. $\text{ctn}\left(\dfrac{C + A}{2}\right) = \tan\dfrac{B}{2}$.
12. $\sec\dfrac{C}{2} = \csc\left(\dfrac{A + B}{2}\right)$.

14. Tables of natural and logarithmic trigonometric functions.* In Art. 7 of the preceding chapter, the trigonometric functions of some special angles were computed. By advanced methods, beyond the scope of this book, tables have been calculated to various degrees of accuracy, giving the numerical values and the logarithmic values of the sine,

* A knowledge of the theory and use of logarithms is presupposed. If the student has not studied logarithms, or if a review of the subject is deemed desirable, Chapter VII should be taken up before continuing with this chapter.

cosine, tangent, and cotangent of any desired acute angle. For ordinary applications, four or five-place tables* are commonly employed. However, in extended surveys, in astronomy, and for work done by instruments of high precision, tables computed to six, seven, or more decimal places are available. In the discussion following, the use of a five-place table is assumed.

The natural or numerical values as well as the logarithmic values of the trigonometric functions of any acute angle expressed in degrees and an integral number of minutes may be read directly from tables marked **The Natural Trigonometric Functions** and **The Logarithmic Trigonometric Functions** respectively. For angles less than 45°, the columns are read downward with the degrees and the names of the functions at the top and the minutes on the left; for angles greater than 45°, the columns are read upward with the degrees and the names of the functions at the bottom and the minutes on the right. To find a function of an angle which does not reduce to an integral number of minutes, the process of **interpolation** is employed. Except for angles near 0° and 90°, interpolation between any two consecutive values in a given table gives a result correct to the same number of places of that table. The method of interpolation is best understood from examples.

Example 1. Find $\sin 27° 34'.7$.

From the table of natural functions, $\sin 27° 34' = 0.46278$.

$$\text{Hence, } \sin 27° 34'.7 = 0.46278 + \tfrac{7}{10}(\sin 27° 35' - \sin 27° 34')$$
$$= 0.46278 + 0.00018 = 0.46296.$$

It is customary when interpolating to disregard the decimal point. The difference for 1′, called the **tabular difference**, would then be

* The exercises in this book have been solved by the use of five-place tables. If four-place tables are to be used, change all five significant digits to the nearest number of four significant digits and all angles to the nearest minute. For example, 14.822, 598.75, and 46° 27′.7 would be replaced by 14.82, 598.8, and 46° 28′ respectively. Cf. Art. 66.

read as 26 instead of 0.00026 and the **correction** for 0.7 as 18 instead of 0.00018.

Example 2. Given $\cos B = 0.52726$, to find the acute angle B.

In the table of natural functions, the given cosine lies between 0.52745 and 0.52720 which are the cosines of $58°\,10'$ and $58°\,11'$ respectively. Therefore B must have a value greater than $58°\,10'$ and less than $58°\,11'$. The actual difference between $\cos B$ and $\cos 58°\,10'$ is -19. The tabular difference is -25. Hence, the actual difference divided by the tabular difference gives the correction of $\dfrac{-19}{-25}$ or 0.8 as the decimal part of a minute. Therefore, $B = 58°\,10'.8$.

Example 3. Find $\log \operatorname{ctn} 62°\,46'\,39''$.

From the table of logarithmic functions, $\log \operatorname{ctn} 62°\,46' = 9.71153 - 10$. The tabular difference is -32, hence the correction is $\frac{39}{60}$ of -32 or -21. Therefore, $\log \operatorname{ctn} 62°\,46'\,39'' = 9.71132 - 10$.

The student should note that the sines and cosines of all acute angles, the tangents of all acute angles less than 45°, and the cotangents of all acute angles greater than 45° are numerically less than unity. Hence the characteristics of their logarithms are negative, but the -10 has been omitted in the table for simplicity of printing. The -10 should be written whenever such a logarithm is used.

Example 4. Given $\log \tan \theta = 0.61207$, to find the acute angle θ.

In the table of logarithmic functions, the given tangent lies between 0.61192 and 0.61246 which are the tangents of $76°\,16'$ and $76°\,17'$ respectively. Since the actual difference is 15 compared to the tabular difference of 54, the correction is $\frac{15}{54}'$ or $0.3'$. Hence, $\theta = 76°\,16'.3$. The correction could have been obtained directly from the table of proportional parts.

Example 5. Evaluate $\dfrac{\operatorname{ctn}^2 45°\,28'.1 \sqrt{\sin\,18°\,17'\,38''}}{(\cos 34°\,20'.4)^{0.3}}$ correct to five significant figures.

$$\text{Let } x = \frac{\operatorname{ctn}^2 45°\,28'.1 \sqrt{\sin 18°\,17'\,38''}}{(\cos 34°\,20'.4)^{0.3}}.$$

Taking the logarithm of both members,

$$\log x = 2 \log \operatorname{ctn} 45°\,28'.1 + \tfrac{1}{2} \log \sin 18°\,17'\,38'' - 0.3 \log \cos 34°\,20'.4.$$

Before using the tables, make an outline in which every operation is indicated and a place provided for each logarithm or anti-logarithm to be used in the computation. All work should be neatly and systematically arranged. The following arrangement which can be modified to meet the needs of each problem is suggested to the student. The first column indicates the operation, the second gives the original logarithm, the third the logarithm resulting from the operation indicated in the first column, and the fourth any required anti-logarithm.

Indicated operation	original log	derived log	anti-log
2 log ctn 45° 28′.1	9.99290−10	9 98580−10	
½ log sin 18° 17′ 38″	9.49678−10	9.74839−10	
log numerator		19.73419−20	
0.3 log cos 34° 20′.4	9 91682−10	9.97505−10	
log x		9 75914−10	
x			0.57430

$$\begin{array}{r} 9\ 99290-10 \\ 2 \\ \hline 19.98580-20 \end{array} \qquad \begin{array}{r} 2\ )\overline{19.49678-20} \\ 9.74839-10 \end{array} \qquad \begin{array}{r} 9.91682-10 \\ 0\ 3 \\ \hline 2\ 975046-\ 3 \end{array}$$

EXERCISES

Find the values of the sine, cosine, tangent, and cotangent of each of the following angles:

1. 24° 19′.8. **3.** 33° 55′ 14″. **5.** 43° 44′.3.
2. 9° 37′ 43″. **4.** 76° 3′.4. **6.** 81° 28′ 52″.

Find the acute angle A, having given:

7. $\sin A = 0.28273$. **10.** $\text{ctn}\, A = 0.06632$. **13.** $\cos A = 0.98366$.
8. $\cos A = 0.57896$. **11.** $\sin A = 0.85780$. **14.** $\tan A = 4.4003$.
9. $\tan A = 0.93057$. **12.** $\text{ctn}\, A = 1.7517$. **15.** $\text{vers}\, A = 0.42137$.

Find the value of each of the following:

16. log tan 39° 53′.5. **18.** log sin 62° 7′.8.
17. log cos 51° 34′ 19″. **19.** log ctn 26° 41′ 39″.

20. $\log \sec 14^\circ 26'.2$.

21. $\log \csc 81^\circ 11' 15''$.

22. $\log \sin 48^\circ 49'.1$.

23. $\log \cos 7^\circ 56' 52''$.

24. $\log \tan 73^\circ 16'.6$.

25. $\log \operatorname{ctn} 68^\circ 9'.3$.

Find the acute angle θ, having given:

26. $\log \operatorname{ctn} \theta = 0.35887$.

27. $\log \sin \theta = 9.47793 - 10$.

28. $\log \cos \theta = 9.27099 - 10$.

29. $\log \tan \theta = 0.34712$.

30. $\log \csc \theta = 0.77706$.

31. $\log \sec \theta = 0.84305$.

32. $\log \cos \theta = 9.59002 - 10$.

33. $\log \tan \theta = 9.78131 - 10$.

34. $\log \sin \theta = 9.93482 - 10$.

35. $\log \operatorname{ctn} \theta = 9.61798 - 10$.

Find the acute angle B, having given:

36. $\sin B = \dfrac{\operatorname{ctn} 73^\circ 32'.4}{\sin 58^\circ 39' 44''}$.

37. $\cos B = \dfrac{\tan 18^\circ 5'.5}{\cos 49^\circ 12'.1}$.

38. $\tan B = \dfrac{\sec 53^\circ 12'.9}{\tan 36^\circ 55' 7''}$.

39. $\operatorname{ctn} B = \dfrac{0.41096 \tan^2 32^\circ 46' 21''}{\cos^3 56^\circ 19'.3}$.

Evaluate to five significant figures:

40. $\dfrac{\sin 29^\circ 44'.2 + \cos 19^\circ 40'.9}{\operatorname{ctn} 30^\circ 53'.6 - \tan 69^\circ 28'.6}$.

41. $\dfrac{\operatorname{ctn} 17^\circ 24'.2 + \tan 82^\circ 47'.8}{\sin 17^\circ 41'.1 - \cos 36^\circ 30'.4}$.

42. $\dfrac{\sec 14^\circ 39'.6 \tan 64^\circ 48'.7}{\sin 77^\circ 51' 19''}$.

43. $\dfrac{\sin 29^\circ 12'.3 + \operatorname{ctn} 51^\circ 29' 11''}{\sqrt{\cos 68^\circ 16'.7}}$.

44. $\dfrac{\csc 38^\circ 19'.8 \sqrt[3]{\tan 61^\circ 49'.5}}{\sec^2 47^\circ 28' 44''}$.

45. $\sqrt[3]{\dfrac{\operatorname{ctn} 44^\circ 57'.5}{\sin 67^\circ 9' 17'' - \cos^2 81^\circ 33'.2}}$.

15. Solution of right triangles. By means of geometry, any triangle may in general be constructed when any three parts are given, one of which is a side. The remaining parts may then be obtained to a limited degree of accuracy by direct measurement from the figure. By trigonometry, however, the numerical values of the unknown parts can be computed accurately, by means of the trigonometric functions, to as

many significant figures as the number of places in the table used. This process is called **solving** the triangle.

Consider the right triangle ABC in which a, b, c denote the lengths of the sides opposite the acute angles A, B, and the right angle C, respectively. In order to solve this triangle, the following general directions are suggested:

(1) *Construct the triangle accurately to some convenient scale with ruler, protractor, and compasses. Find the unknown parts by direct measurement from the figure. These estimated values should serve as a rough check on the results found by the method of trigonometry.*

(2) *Select for each unknown part the trigonometric function which involves that unknown part and two known parts. The unknown parts can then be calculated by employing either natural functions or logarithms. In either case, the necessary formulas should be written down first, solved for the required parts, and a complete skeleton scheme of the solution made before using the tables. The Pythagorean relation* $c^2 = a^2 + b^2$ *should be avoided unless the given data is easily squared. All work should be neatly and systematically arranged.*

(3) *Finally, a check against numerical errors is necessary. Any formula not used in the solution and containing as many of the computed values as possible may be used for this purpose. For right triangles,* $a^2 = c^2 - b^2 = (c + b)(c - b)$ *is a convenient numerical check. The graphical solution should detect large errors.*

Example 1. Solve the right triangle when $c = 15.007$, and $A = 36° 31'.4$: (a) by natural functions; (b) by logarithms.

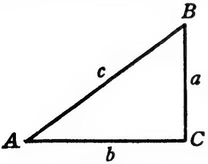


FIG. 20

Given	To find*	Estimated
$c = 15.007$	$a = 8.9314$	$a = 9$
$A = 36° 31'.4$	$b = 12.060$	$b = 12$
$C = 90°$	$B = 53° 28'.6$	$B = 53°$

$$B = 90° - 36° 31'.4 = 53° 28'.6.$$

* To be filled in as the results are found.

(a) *By natural functions:*

$\sin A = \frac{a}{c}$ or $a = c \sin A$. $\cos A = \frac{b}{c}$ or $b = c \cos A$.

$a = 15.007 \times 0.59515 = 8.9314$. $b = 15.007 \times 0.80361 = 12.060$.

Check

$$\frac{b}{a} = \tan B \quad \text{or} \quad b = a \tan B.$$

$$12.060 = 8.9314 \times 1.3503 = 12.060.$$

(b) *By logarithms:*

$a = c \sin A$.

$\log a = \log c + \log \sin A$.

log c	1.17629
log sin A	9.77463−10
log a	0 95092
a	8.9314

$b = c \cos A$.

$\log b = \log c + \log \cos A$.

log c	1 17629
log cos A	9 90505−10
log b	1 08134
b	12.060

Check

$b = a \tan B$.

$\log b = \log a + \log \tan B$.

log a	0.95092
log tan B	0.13042
log b	1.08134

Example 2. Solve the right triangle when $a = 0.82125$ and $b = 0.65700$: (a) by natural functions; (b) by logarithms.

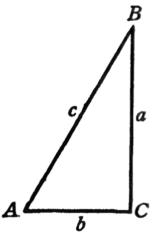


FIG. 21

Given	To find	Estimated
$a = 0.82125$	$A = 51° 20'.4$	$A = 52°$
$b = 0.65700$	$B = 38° 39'.6$	$B = 38°$
$C = 90°$	$c = 1.0517$	$c = 1.05$

(a) *By natural functions:*

$$\tan A = \frac{a}{b} = \frac{0.82125}{0.65700} = 1.2500. \qquad \frac{a}{c} = \sin A \quad \text{or}$$

$$A = 51° 20'.4. \qquad c = \frac{a}{\sin A}.$$

$$B = 90° - 51° 20'.4 \qquad c = \frac{0.82125}{0.78087} = 1.0517.$$
$$= 38° 39'.6.$$

Check

$$\frac{b}{c} = \sin B \quad \text{or} \quad b = c \sin B.$$

$$0.65700 = 1.0517 \times 0.62470 = 0.65700.$$

(b) *By logarithms:*

$$\tan A = \frac{a}{b}. \qquad c = \frac{a}{\sin A}.$$

$$\log \tan A = \log a - \log b. \qquad \log c = \log a - \log \sin A.$$

$\log a$	9 91448–10
$\log b$	9 81757–10
$\log \tan A$	0 09691
A	51° 20′.4

$\log a$	9.91448–10
$\log \sin A$	9.89258–10
$\log c$	0.02190
c	1 0517

$$B = 90° - 51° 20'.4 = 38° 39'.6.$$

Check

$$b = c \sin B.$$
$$\log b = \log c + \log \sin B.$$

$\log c$	0.02190
$\log \sin B$	9.79567–10
$\log b$	9.81757–10

EXERCISES

Solve the following right triangles by logarithms and also find the areas of the starred problems, having given:

1.* $c = 3.7986$, $A = 43° 47' 16''$.

2. $a = 2461.8$, $b = 1975.9$.

3. $b = 88.469$, $B = 64° 10' 39''$.

4. $c = 110.82$, $B = 75° 22'.5$.

5. $b=0.90009$, $A=26° 58'.3$.
6.* $a=3.0958$, $c=7.4265$.
7.* $b=720.93$, $c=817.34$.
8. $a=14.424$, $B=50° 52'.8$.
9. $a=6.0907$, $A=37° 8' 44''$.
10. $c=0.14825$, $a=0.091674$.
11. $b=39.191$, $c=45.357$.
12.* $b=0.091273$, $a=0.052649$.
13.* $b=0.84726$, $B=41° 58'.8$.
14. $a=0.012249$, $B=80° 19' 38''$.
15. $c=7.8201$, $a=1.4327$.
16. $c=0.0013346$, $b=0.0010229$.
17. $b=201.39$, $A=78° 6'.3$.
18.* $a=68.202$, $A=22° 58'.3$.

Solve the following right triangles by natural functions and also find the areas of the starred problems, having given:

19. $c=2.1704$, $A=47° 33'.6$.
20. $a=17.604$, $c=22.005$.
21.* $b=17.616$, $a=7.0464$.
22.* $a=34.096$, $B=72° 11'13''$.
23. $c=14500$, $b=13200$.
24. $b=0.60931$, $B=14° 55'.8$.

Solve the following isosceles triangles, a being one of the equal sides, b the base, and h the altitude:

25. $h = 15.486$, $a=27.096$.
26. $b=7.8962$, $B=99° 15' 38''$.
27. $a=100.94$, $A=56° 49'.8$.
28. $b=0.49766$, $h=0.52029$.
29. $a=17.621$, $b=14.207$.
30. $a=21.714$, $B=86° 19'.8$.
31. $b=10.406$, $B=100° 49'.8$.
32. $h=13.098$, $A=22° 37' 19''$.

33. In a regular octagon, the length of a side is 10.463 in. Find the radius of the circumscribed circle.

34. A regular decagon is inscribed in a circle whose diameter is 1.4346 ft. Find the perimeter and the area of the decagon.

35. Find the area and perimeter of a regular hexagon inscribed in a circle 17.550 cm. in diameter.

In each of the following right triangles, find the length of the perpendicular h from C to AB:

36. $c=60.789$, $A=71° 16' 49''$.
37. $b=2.5429$, $a=1.7632$.
38. $a=0.42937$, $c=0.70084$.
39. $c=317.65$, $B=28° 44'.6$.

16. Solution of oblique triangles by means of right triangles. In oblique triangles, the same notation is used as in right triangles to represent the sides and angles, except that C is no longer a right angle. A general method for solving such triangles consists in dividing the triangle into two right triangles by drawing a perpendicular from a vertex to the opposite side (produced if necessary) and then

solving the resulting right triangles. In all cases, except when three sides are given, the perpendicular can be so drawn that one of the resulting triangles contains two of the given parts. The method is illustrated by the following examples.

Example 1. Solve the oblique triangle when b, c, and A are given.

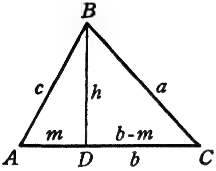


FIG. 22

Given	To find	Estimated
$b =$	$a =$	$a =$
$c =$	$B =$	$B =$
$A =$	$C =$	$C =$

From the vertex B, drop the perpendicular h upon the side b, dividing it into the segments m and $b - m$ respectively. Then, in the right triangle ADB, find h and m; and in the right triangle BDC, using h and $b - m$, find C. Hence B and a can be easily found.

Example 2. Solve the oblique triangle when a, b, and c are given.

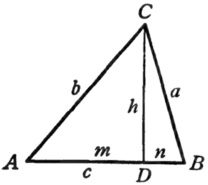


FIG. 23

Given	To find	Estimated
$a =$	$A =$	$A =$
$b =$	$B =$	$B =$
$c =$	$C =$	$C =$

Drop the perpendicular h upon any side c from the vertex of the opposite angle C, dividing it into the segments m and n respectively. Then,

$$h^2 = b^2 - m^2 = a^2 - n^2, \tag{1}$$

or

$$m^2 - n^2 = b^2 - a^2. \tag{2}$$

$$\therefore m - n = \frac{(b - a)(b + a)}{m + n} = \frac{(b - a)(b + a)}{c}, \tag{3}$$

since

$$m + n = c. \tag{4}$$

Adding and subtracting (3) and (4), there results

$$m = \tfrac{1}{2}\left[c + \frac{(b-a)(b+a)}{c}\right], \tag{5}$$

and

$$n = \tfrac{1}{2}\left[c - \frac{(b-a)(b+a)}{c}\right] \tag{6}$$

If either m or n is negative, the point D is on the line AB produced.

Having found m and n, the angles A and B can be obtained from the right triangles ADC and CDB respectively. Then the third angle C can be found from the relation $A + B + C = 180°$.

EXERCISES

Solve the following oblique triangles having given:

1. $A = 54° 46'.7$, $B = 75° 13'.2$, $c = 3.5068$.
2. $a = 28.632$, $b = 33.007$, $C = 48° 57'.4$.
3. $a = 180.97$, $c = 358.36$, $C = 69° 41'.3$.
4. $a = 2.8371$, $c = 2.4865$, $B = 86° 9' 34''$.
5. $a = 62.803$, $b = 97.179$, $c = 76.624$.
6. $B = 62° 25' 17''$, $C = 71° 3' 50''$, $c = 0.19444$.
7. $b = 1473.6$, $c = 1120.9$, $B = 83° 30'.5$.
8. $a = 2.4758$, $b = 2.8631$, $c = 1.9967$.
9. $a = 24.067$, $c = 13.985$, $A = 58° 26' 49''$.
10. $a = 0.40009$, $b = 0.28486$, $c = 0.38293$.
11. $b = 0.094362$, $c = 0.11043$, $A = 77° 0'.6$.
12. $A = 35° 19' 52''$, $C = 80° 47' 5''$, $a = 121.87$.

17. Terms occurring in trigonometric problems.

The **vertical line (plumb line)** at any point on the earth's surface is the line joining that point to the center of the earth.

A **vertical plane** at any point is a plane which contains the vertical line at that point.

A **horizontal line** at any point is the line which is perpendicular to the vertical line at that point.

The **horizontal plane** at any point is the plane which is perpendicular to the vertical line at that point.

A **horizontal** (or **vertical**) **angle** is one lying in a horizontal (or vertical) plane.

An **engineer's transit** is an instrument for measuring horizontal and vertical angles.

The **angle of elevation** (or **depression**) of an object above (or below) the horizontal plane of the observer is the vertical

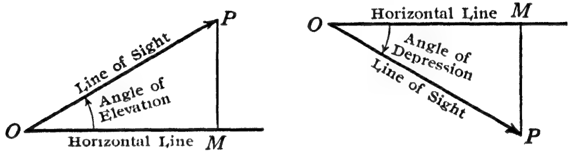


FIG. 24

angle between the line of sight and a horizontal line through the observer's eye (Fig. 24).

The **horizontal** (or **vertical**) **distance** between two points is the distance from one of the two points to the vertical line (or horizontal plane) through the other. Thus, in Fig. 24, OM is the horizontal and MP the vertical distance from O to P.

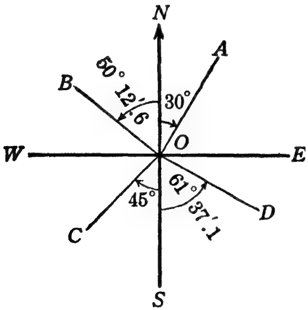


FIG. 25

The **angle subtended** by a line is the angle obtained by joining the point of observation to the ends of the line.

The **bearing** of a line is the horizontal acute angle which the line makes with the north and south line. Thus, in Fig. 25, if O be the point of observation, the bearing of OA is $N\,30°\,E$; of OB, $N\,50°\,12'.6\,W$; of OC, $S\,45°\,W$; and of OD, $S\,61°\,37'.1\,E$.

18. Applications. Some applications of the trigonometric functions to various practical problems such as finding directions, distances, heights, widths, and areas will now be considered. The data for such problems are derived from observations made with instruments of various degrees of precision. Therefore, the student should not carry out the computations in the solutions of these problems to a greater degree of accuracy than that of the given data. Throughout this book, *all linear measurements are assumed accurate to five significant figures and all angular measurements correct to tenths of a minute or to seconds.*

The general directions given in connection with the solution of right triangles also apply here. For checking, the graphical solution and the student's sense of values should be sufficient. Before doing any of the numerical computation, it is important that a general algebraic solution should be first obtained.

Example 1. The angle of elevation of the top of a tower $114\frac{1}{4}$ ft. high, situated on one bank of a river, is $36° 12' 33''$ from a point on the opposite bank. Find the width of the river.

Given	To find	Estimated
$h = 114.25$ ft.	$x = 156.05$ ft.	$x = 155$ ft.
$\alpha = 36° 12' 33''$.		

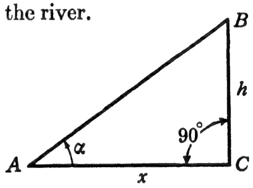


Fig. 26

In the right triangle ACB,

$$\frac{x}{h} = \text{ctn}\,\alpha, \qquad (1)$$

or

$$x = h\,\text{ctn}\,\alpha. \qquad (2)$$

Taking the logarithm of both members of (2),

$$\log x = \log h + \log \text{ctn}\,\alpha.$$

$\log h$	2 05786
$\log \text{ctn}\,\alpha$	0.13541
$\log x$	2.19327
x	156.05

Example 2. At a certain point the angle of elevation of a mountain peak is 39° 41′.8. At a second point 1500 ft. farther away int he same horizontal plane, its angle of elevation is 28° 17′.4. Find the height of the peak above the horizontal plane, and the horizontal distance from the first point of observation to the peak, assuming that the two points of observation and the peak are in the same vertical plane.

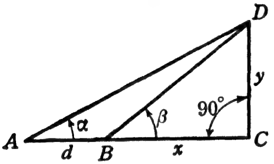


Fig. 27

Given	To find	Estimated
$\alpha = 28° 17'.4$	$y = 2295.7$ ft.	$y = 2300$ ft.
$\beta = 39° 41'.8$	$x = 2765.5$ ft.	$x = 2800$ ft.
$d = 1500$ ft.		

In the right triangle ACD,

$$\text{ctn}\,\alpha = \frac{d + x}{y}, \tag{1}$$

and in the right triangle BCD,

$$\text{ctn}\,\beta = \frac{x}{y}. \tag{2}$$

Subtracting (2) from (1),

$$\text{ctn}\,\alpha - \text{ctn}\,\beta = \frac{d}{y}, \tag{3}$$

or

$$y = \frac{d}{\text{ctn}\,\alpha - \text{ctn}\,\beta}. \tag{4}$$

From the table of natural functions,

$$\begin{aligned} \text{ctn}\,\alpha &= 1.8580, \\ \text{ctn}\,\beta &= 1.2046. \\ \hline \therefore \text{ctn}\,\alpha - \text{ctn}\,\beta &= 0.6534. \end{aligned}$$

Taking the logarithm of both members of (4),

$$\log y = \log d - \log\,(\text{ctn}\,\alpha - \text{ctn}\,\beta).$$

$\log d$	3.17609
$\log\,(\text{ctn}\,\alpha - \text{ctn}\,\beta)$	9 81518−10
$\log y$	3.36091
y	2295 7

From (2), $x = y \operatorname{ctn} \beta.$

Then, $\log x = \log y + \log \operatorname{ctn} \beta.$

$\log y$	3.36091
$\log \operatorname{ctn} \beta$	0 08086
$\log x$	3.44177
x	2765.5

Example 3. A building and a tower stand on the same horizontal plane. The angles of depression of the top and bottom of the building viewed from the top of the tower 120 ft. high are 24° 46′.8 and 56° 27′.2 respectively. Find the height of the building.

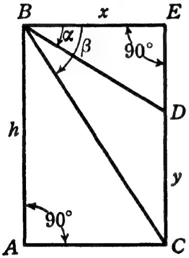


Fig. 28

Given	To find	Estimated
$\alpha = 24° 46'.8$	$y = 83.274$ ft.	$y = 83$ ft.
$\beta = 56° 27'.2$		
$h = 120$ ft.		

In the right triangle BEC,

$$\tan \beta = \frac{h}{x}, \tag{1}$$

and in the right triangle BED,

$$\tan \alpha = \frac{h - y}{x}. \tag{2}$$

Subtracting (2) from (1),

$$\tan \beta - \tan \alpha = \frac{y}{x}, \tag{3}$$

or

$$y = x(\tan \beta - \tan \alpha). \tag{4}$$

Upon substituting from (1) $x = \dfrac{h}{\tan \beta}$ in (4),

$$y = \frac{h(\tan \beta - \tan \alpha)}{\tan \beta}. \qquad (5)$$

From the table of natural functions,

$$\begin{aligned} \tan \beta &= 1.5082, \\ \tan \alpha &= 0.46164. \\ \hline \therefore \tan \beta - \tan \alpha &= 1.04656 \\ &= 1.0466, \text{ to five significant figures.} \end{aligned}$$

Taking the logarithm of both members of (5),

$$\log y = \log h + \log(\tan \beta - \tan \alpha) - \log \tan \beta.$$

$\log h$	2.07918
$\log(\tan \beta - \tan \alpha)$	0 01978
log numerator	2 09896
$\log \tan \beta$	0 17845
$\log y$	1 92051
y	83.274

Example 4. An open belt connects two pulleys of radii 11 in. and 7 in. respectively. If the distance between their centers is 23 in., find the length of the belt.

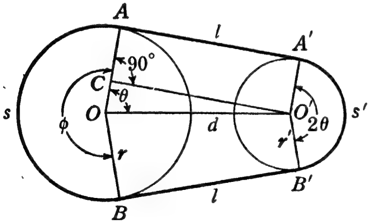


Fig. 29

Given	To find
$r = 11$ in.	Length
$r' = 7$ in.	of belt =
$d = 23$ in.	103.25 in.

In the right triangle $OO'C$,

$$\cos \theta = \frac{OC}{d} = \frac{4}{23} = 0.17391,$$

from which

$$\theta = 79^\circ\, 59'.1.$$

Then,

$$2\,\theta = 159^\circ\, 58'.2 \quad \text{and} \quad \phi = 360^\circ - 159^\circ\, 58'.2 = 200^\circ\, 1'.8.$$

Converting to radians, using Table II,

$$2\,\theta = 2.79200 \text{ rad.} \quad \text{and} \quad \phi = 3.49118 \text{ rad.}$$

Hence,

$$\begin{array}{rcrcr}
s = & & r\phi = 11 \times 3.49118 & = & 38.403 \text{ in.,} \\
s' = & & r' \cdot 2\,\theta = 7 \times 2.79200 & = & 19.544 \text{ in.,} \\
2\,l = & & 2 \cdot O'C = 2\,\sqrt{23^2 - 4^2} & = & 45.299 \text{ in.} \\
\hline
 & & \therefore \text{ Length of belt} & = & 103.246 \text{ in.} \\
 & & & = & 103.25 \text{ in.}
\end{array}$$

EXERCISES

1. Find the angle of elevation of the sun if a tower $275\frac{3}{4}$ ft. high casts a shadow $322\frac{1}{4}$ ft. long.

2. The angle of depression of a boat from the top of a cliff 225.65 ft. high is $29^\circ\, 41'\, 29''$. Find the distance of the boat from the foot of the cliff.

3. A tunnel into the earth descends at an angle of depression of $16^\circ\, 30'.5$. What is the vertical distance between two points which are $95\frac{1}{8}$ yds. apart along the tunnel?

4. Find the area of a parallelogram whose sides are 24.632 cm. and 31.708 cm., the acute angle between them being $49^\circ\, 7'.8$.

5. What angle does a rafter make with the horizontal if it has a rise of $7\frac{1}{3}$ ft. in a run of $13\frac{1}{8}$ ft.?

6. A ship is sailing at the rate of 25 mi. per hr. in a direction $S\ 79^\circ\, 18'.7\ W$. At what rate is the ship going southward?

7. A wedge measures $10\frac{3}{4}$ in. along each side and the base is $1\frac{1}{2}$ in. wide. Find the angle at the vertex.

8. Two ships leave the same dock at the same time in directions $N\ 23^\circ\, 17'.9\ E$ and $S\ 66^\circ\, 42'.1\ E$ at rates of 17 and 22 mi. per hr. Find their distance apart after 2 hr. 15 min. 45 sec.

9. The horizontal distance between the two extreme positions of the end of a pendulum 27 in. long is 7 in. Through what angle does it swing? How far does the end travel from one extreme position to the other?

10. A ladder 20 ft. long is leaning against the side of a house, and makes an angle of 34° 30′.0 with the wall. Find its projections upon the wall and upon the ground.

11. A tower 147.64 ft. high is situated on the bank of a river. The angle of depression of an object on the opposite bank is 31° 49′.7. Find the breadth of the river.

12. If an automobile weighing 3200 lbs. is standing on a road which slopes 12°, what force tends to pull it down the hill?

13. At a certain point the angle of elevation of the top of a hill is 30° 53′.6. On moving 200 ft. directly away and in the same horizontal plane, the angle of elevation was observed to be 20° 31′.4. How high is the hill?

14. To an observer the angle of elevation of the top of a church 260.33 ft. away is 22° 57′.7, and the angle subtended by the spire above it is 10° 13′.2. Find the height of the spire.

15. In the side of a hill which slopes upward at an angle of 24°, a tunnel is bored sloping downward at an angle of 15° with the horizontal. What is the vertical distance to the surface of the hill from a point 175 ft. down the tunnel?

16. From the top of a cliff 196 ft. high the angles of depression of two boats, which are due east of the observer, are 17° 24′ 35″ and 64° 51′ 19″. Find the distance between the boats.

17. From a point 25 ft. above the surface of the water, the angle of elevation of the top of an observation tower standing at the edge of the water is 37° 29′ 55″, while the angle of depression of its image in the water is 49° 54′ 18″. Find the height of the tower.

Note. Any point on the image appears to the observer to be as far below the water surface as the corresponding point on the object is above.

18. The angle of elevation of the top of a pole from the top of a house 42 ft. high is 14° 26′ 9″. At the bottom of the house it is 23° 21′ 33″. Find the height of the pole.

19. Two trees of equal height stand on opposite sides of a roadway 150 ft. wide. At a certain point in the road between the trees, the angles of elevation of their tops are 42° 18′.7 and 22° 42′.8 respectively. Find the height of the trees.

20. At a certain point the angle of elevation of the top of the Washington Monument 555 ft. high was found to be 56° 57′.5.

How far back in the same vertical plane must the observer move in order that the angle of elevation may be $31^\circ\ 7'.1$?

21. At a point A, north of a tower, the angle of elevation of the top of the tower is 60°. At another point B, 200 ft. west of A, the angle of elevation is 30°. Find the height of the tower.

22. From the top of a tower 300 ft. high the angle of depression of a point X due south is 45°, while the angle of depression of a point Y due east of X is 30°. Find the distance between X and Y.

23. At a point south of a hill the angle of elevation of the top is $50^\circ\ 38'.9$, and at a point 500 ft. directly east of the first point the angle of elevation is $44^\circ\ 9'.2$. What is the height of the hill?

24. An open belt connects two pulleys of radii 7 ft. and 1 ft. respectively. If the distance between their centers is 12 ft., find the length of the belt.

25. The diameters of two wheels are 4 ft. and 12 ft. respectively, and the distance between their centers is 12 ft. Find the length of the open belt which connects them.

26. Using the same values as in Ex. 25, find the length of the belt when crossed.

27. Two pulleys of radii 5 ft. and $1\frac{1}{2}$ ft. respectively are connected by a crossed belt. If the centers of the pulleys are 10 ft. apart, find the length of the belt.

28. Using the same values as in Ex. 27, find the length of the belt when open.

GENERAL EXERCISES

1. Show that in any triangle ABC, $\tan\left(\dfrac{A+B}{2}\right) = \operatorname{ctn}\dfrac{C}{2}$.

2. A regular pyramid, with a square base, has a lateral edge 22.764 cm. long, and a side of its base is 18.068 cm. Find the inclination of the face of the pyramid to the base.

3. Solve the oblique triangle ABC having given: $A = 58^\circ\ 39'.7$, $C = 69^\circ\ 57'.2$, and $b = 11.096$.

4. The sides of a triangle are 26.838, 35.257, and 35.257 respectively. Find the angles.

5. Given $\tan\theta = \dfrac{\sqrt{\sin 78^\circ\ 44'\ 13''}}{\sqrt[3]{\sec 9^\circ\ 7'.7}}$, find the acute angle θ.

6. Find the diameter of a circle inscribed in an equilateral triangle whose perimeter is 57.228 in.

7. From the top of a hill I observe that the angles of depression of two successive milestones in the horizontal plain below and in a straight line before me, are $18° 49' 7''$ and $10° 6' 44''$. Find the height of the hill.

8. Evaluate to five significant figures:

$$\frac{\sqrt{\cos 75° 22'.3} - \sin 15° 34'.4}{[\tan 45° 52'.7]^{0.3}}.$$

9. If $\sin(2x - \pi) = \cos\left(\frac{x}{3} + \frac{\pi}{6}\right)$, find a value of x.

10. Two straight stretches of railway, if extended, would meet at a point making an angle of $45° 17'.6$. These two stretches are to be connected by means of a circular arc of radius 4500 ft. Find the distance from the point of tangency to the point of intersection.

11. At the foot of a hill the angle of elevation of its summit is observed to be $28° 31'.4$. After ascending the hill $248\frac{1}{3}$ ft., up a slope of $15°$ inclination, the angle of elevation of its summit is found to be $39° 47'.9$. Find the height of the hill if the two points of observation and the summit are in the same vertical plane.

12. An open belt connects two pulleys of diameters 6 in. and 14 in. respectively. If the distance between their centers is 15 in., find the length of the belt.

13. Using the same values as in Ex. 12, find the length of the belt when crossed.

14. From the top of a tower the angle of depression of a point B due south is $18° 59'.3$, while the angle of depression of a point C, 250 ft. due east of B, is $13° 33'.8$. Find the height of the tower.

15. Evaluate to five significant figures:

$$\sqrt[5]{\frac{\sin 42° 30'.5}{\operatorname{ctn} 19° 21' 33'' - \tan 56° 4'.4}}.$$

16. Solve the isosceles triangle whose altitude is 20.098, one of the equal sides being 29.143.

17. At a distance b from the foot of a tower, the angle of elevation B of the top of the tower is the complement of the angle of elevation

of the top of a flagstaff on top of the tower. Show that the length of the flagstaff is

$$\frac{b(1 - \tan^2 B)}{\tan B}.$$

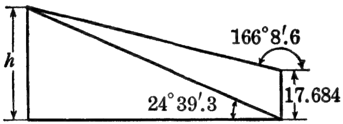
18. A ladder 24 ft. long is resting against a wall at an angle of 65° 14′ 49″. If the foot is drawn away $33\frac{3}{4}$ in., how far down the wall will the top of the ladder fall?

19. Find the angle between the diagonal of a cube and one of the diagonals of a face which meets it.

20. A hill slopes at an angle of 26° 30′.0 with the horizontal. A path leads up it making an angle of 38° 49′.9 with the line of steepest slope. Find the inclination of the path with its horizontal projection.

21. From the top of a tower 420 ft. high the angles of depression of two objects were found to be 54° 19′.7 and 46° 54′.4. Find the distance between the objects, assuming that they are in the same vertical plane as the point of observation and in the same horizontal plane as the foot of the tower.

22.



Find h.

23. The sides of a triangle are 2.1469, 3.8323, and 4.0026 respectively. Find the smallest angle.

24. Evaluate to five significant figures:

$$\frac{\log 0.076925 + \operatorname{ctn} 56° \, 39'.1}{\frac{\pi}{3} - \log \sin 29° \, 18'.4}.$$

25. Find the volume and lateral area of a right circular cone if the base is 12 in. in diameter and the vertical angle is 69° 19′.2.

26. A pyramid has a base 24 in. square and each triangular face makes an angle of 78° 2′.1 with the base. Find the lateral area and the volume.

27. From the top and bottom of a $21\frac{3}{8}$ ft. wall the angles of elevation of a pole in the same horizontal plane were found to be 36° 49′.7 and 43° 18′.2 respectively. Find the height of the pole.

28. In surveying a mine, a man measures a length $XY = 375$ ft. due west with a dip of $8° 19'.0$, then a length $YZ = 245$ ft. due south with a dip of $6° 54'.0$. How much deeper is Z than X?

29. At a certain point A the angle of elevation of a mountain peak is α; at a point B that is a miles further away in the same horizontal plane its angle of elevation is β. If h represents the distance the peak is above the plane and x the horizontal distance the peak is from A, show that:

$$x = \frac{a \tan \beta}{\tan \alpha - \tan \beta} \quad \text{and} \quad h = \frac{a \tan \alpha \tan \beta}{\tan \alpha - \tan \beta}.$$

30. At a point south of a hill 300 ft. high the angle of elevation of the top is $49° 46'.6$, and at a point directly west of the first point the angle of elevation is $42° 21'.7$. Find the distance between the points.

31. Solve the oblique triangle ABC having given: $a = 6.9608$, $c = 8.7653$, and $B = 49° 16'.9$.

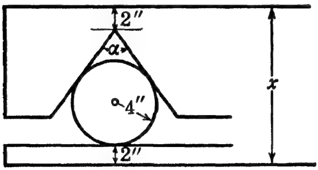
32. From a mountain 3 mi. high the angle of depression of the most distant visible object is $2° 13' 50''$. Find the radius of the earth.

33. Prove that in any right triangle ABC, $\tan \frac{A}{2} = \sqrt{\frac{c-b}{c+b}}$.

34. How high above the earth must one be to see a point on the surface 75 mi. away, assuming that the radius of the earth is 3963.3 mi.?

35. At two points B and C, 500 yds. apart on a straight horizontal road, the summit of a hill is observed. At B it is due north, with an elevation of $42° 11'.2$ and at C it is due east, with an elevation of $31° 44'.7$. Find the height of the hill.

36.



Given $\alpha = 66° 12' 24''$, find x.

37. The radius of a circle is 60.480 cm. Find the difference between the lengths of the arc and chord intercepted by a central angle of $44° 16' 26''$.

38. A statue 13 ft. high subtends an angle of $13° 37'.3$ at a point on the ground where the angle of elevation of the pedestal upon which it stands is $31° 22'.7$. How high is the pedestal?

39. At its lowest and highest positions, respectively, the ball of a pendulum is 3 in. and 5 in. above the floor. Calculate the length of the arc through which the ball oscillates if the point of suspension is 14 in. above the floor.

40. From two successive milestones on a straight horizontal road, the angles of elevation of a balloon are $44° 27'.2$ and $33° 43'.7$. Find the height of the balloon above the road if it is directly above the road at a point between the milestones.

CHAPTER III

RELATIONS AMONG THE TRIGONOMETRIC FUNCTIONS OF RELATED ANGLES

19. Introduction. The next problem to be considered is that of using the tables of trigonometric functions to find the functions of angles other than acute angles. For example, the general methods of solving oblique triangles frequently require the evaluation of such expressions as $\sin 117° 28'.2$, $\cos 162° 48' 26''$, $\log \sin 119° 16'.4$, or these functions of other obtuse angles. In other problems the functions of negative and of large angles are needed, hence it is necessary to be able to extend the use of tables to angles of all magnitudes.

Since any angle can be represented by $n \cdot 90° \pm \theta$, n being any integer, positive or negative, and θ an acute angle, this problem becomes one of expressing the functions of $n \cdot 90° \pm \theta$ in terms of functions of θ. The developments of the following articles show how to find the required relationships when θ is not only acute, but also when it is located in some particular quadrant or quadrants, the results found, however, being true for all values of θ.

20. Functions of $-\theta$ in terms of functions of θ. Consider any positive angle θ, illustrations of possible choices, one in each quadrant, being shown in Fig. 30. Construct the corresponding numerically equal negative angle. Let $P(x, y)$ be any point on the terminal side of θ and r its distance from O. Take $P_1(x_1, y_1)$ on the terminal side of the negative angle so that $OP_1 = OP$, and let r_1 be the distance of P_1 from O. Perpendiculars from P and P_1 to the x-axis at M and M_1, respectively, give the congruent triangles OMP and OM_1P_1. If the terminal side of the angle is so

chosen that it very obviously does not bisect the quadrant in which it falls, it will be easier to recognize the corresponding sides of the congruent triangles.

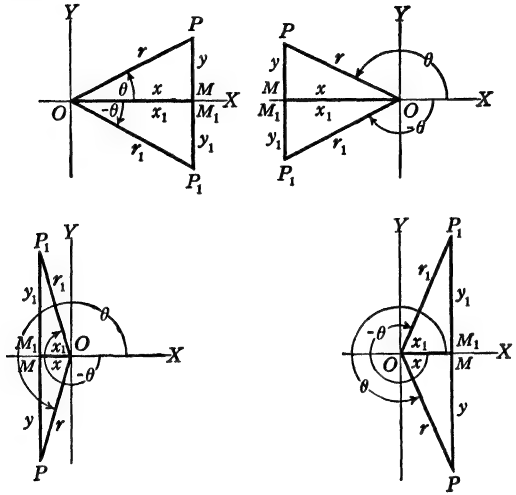


FIG. 30

For each figure

$$r_1 = r, \quad x_1 = x, \quad y_1 = -y,$$

these being the corresponding sides of the congruent triangles. The equation, $y_1 = -y$, means that y_1 and y are two directed lines of the same length but of opposite directions. The equation could just as well have been written $y = -y_1$.

Applying the definition of the sine to $-\theta$, $\sin(-\theta) = \dfrac{y_1}{r_1}$.

By substitution from the equations above, $\dfrac{y_1}{r_1} = \dfrac{-y}{r} = -\dfrac{y}{r}$,

and as $\dfrac{y}{r} = \sin\theta$, $\sin(-\theta) = -\sin\theta$. The other functions may be treated similarly. The complete process is shown below.

In each quadrant

$$\sin(-\theta) = \frac{y_1}{r_1} = \frac{-y}{r} = -\frac{y}{r} = -\sin\theta,$$

$$\cos(-\theta) = \frac{x_1}{r_1} = \frac{x}{r} = \cos\theta,$$

$$\tan(-\theta) = \frac{y_1}{x_1} = \frac{-y}{x} = -\frac{y}{x} = -\tan\theta,$$

and using the reciprocal relations with the above,

$$\csc(-\theta) = -\csc\theta, \quad \sec(-\theta) = \sec\theta, \quad \operatorname{ctn}(-\theta) = -\operatorname{ctn}\theta.$$

By a similar proof these relations can be shown to be valid for all other values of the angle θ, hence they are identities.

EXERCISES

From a figure, derive expressions in terms of trigonometric functions of θ, for the sine, cosine, and tangent of $-\theta$ when θ is defined as below:

1. $90° < \theta < 180°$. **2.** $270° < \theta < 360°$.

3. Prove the relations of Art. 20, using the directed lines OM, MP, OP, etc. in place of x, y, r, etc.

Express the trigonometric functions of the following angles as functions of positive angles:

4. $-153°$. **6.** $-\frac{\pi}{3}$. **8.** $-599° 17' 58''$.

5. $-217°$. **7.** $-193° 18' 16''$. **9.** -0.769.

Find the values of the sine, cosine, tangent, and cotangent of the following angles (results correct to five places):

10. $-82° 15'.8$. **12.** $-45° 29'.4$. **14.** $-34° 52' 43''$.

11. $-11° 29'.4$. **13.** $-22° 17'.1$. **15.** $-68° 42' 46''$.

Simplify, expressing results as functions of B:

16. $\sin(-B)\csc(-B) + \cos(-B)\sec(-B)$.

17. $\tan(-B)\cos(-B) + \operatorname{ctn}(-B)\sin(-B)$.

18. $\sin^2(-B) - \cos^2(-B) + \dfrac{\operatorname{ctn}^2(-B)}{\csc^2(-B)}$.

19. Given $\sin C = -\frac{3}{5}$ and $180° < C < 270°$; find the value of $\sin(-C)$, $\cos(-C)$, and $\tan(-C)$.

20. Given $\tan\theta = -\frac{5}{12}$ and $90° < \theta < 180°$; find the numerical value of each of the six functions of $-\theta$.

21. Functions of $90° - \theta$ in terms of functions of θ. Let XOP be any angle θ. As an illustration of the method of derivation, angles in the first and second quadrants have been chosen and represented in Fig. 31. Figures can be

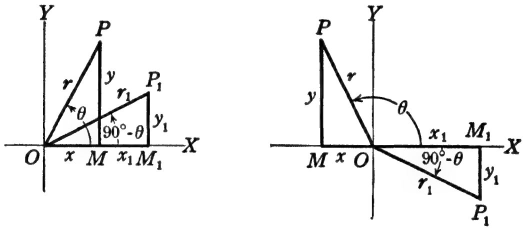


Fig. 31

constructed for angles in other quadrants and the derivation will be identical with that for the angles chosen. Let the corresponding angle, $90° - \theta$, be constructed and represented by XOP_1, and OP_1 be taken equal to OP. Then the triangles OMP and OM_1P_1 are congruent, and it will be easier to recognize the corresponding sides if the terminal side of θ obviously is not a bisector of any quadrant.

For each figure

$$r_1 = r, \quad y_1 = x, \quad x_1 = y,$$

where x, y, r and x_1, y_1, r_1 are associated with P and P_1 respectively. To derive the relations, start with the required function of $90° - \theta$ defined in terms of abscissa, ordinate, and distance; substitute for these their equivalent values as given by the equations above, and replace the new ratio by the proper function of θ.

Then in each quadrant

$$\sin (90^\circ - \theta) = \frac{y_1}{r_1} = \frac{x}{r} = \cos \theta,$$

$$\cos (90^\circ - \theta) = \frac{x_1}{r_1} = \frac{y}{r} = \sin \theta,$$

$$\tan (90^\circ - \theta) = \frac{y_1}{x_1} = \frac{x}{y} = \text{ctn}\, \theta,$$

and from the reciprocal relations

$$\begin{aligned} \csc (90^\circ - \theta) &= \sec \theta, \\ \sec (90^\circ - \theta) &= \csc \theta, \\ \text{ctn}\, (90^\circ - \theta) &= \tan \theta. \end{aligned}$$

Like the relations of the preceding article, these are identical equations since they can be shown to be true for all values of the angle θ.

22. Functions of $90^\circ + \theta$ in terms of functions of θ. Let XOP be any angle θ. To illustrate the method, angles have been chosen in the first and fourth quadrants and represented in Fig. 32. Figures can be constructed for

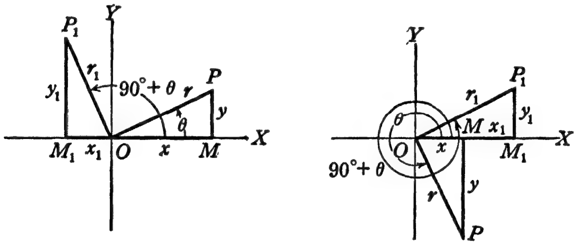


Fig. 32

angles in the other quadrants and the derivation and results will be identical with that for the angles chosen. Let the corresponding angle, $90^\circ + \theta$, be constructed and represented by XOP_1, and let $OP_1 = OP$. The general plan of procedure is the same as that used for $90^\circ - \theta$.

From the congruent triangles OMP and OM_1P_1

$$r_1 = r, \quad x_1 = -y, \quad y_1 = x.$$

Then in each quadrant

$$\sin (90° + \theta) = \frac{y_1}{r_1} = \frac{x}{r} = \cos \theta,$$

$$\cos (90° + \theta) = \frac{x_1}{r_1} = \frac{-y}{r} = -\frac{y}{r} = -\sin \theta,$$

$$\tan (90° + \theta) = \frac{y_1}{x_1} = \frac{x}{-y} = -\frac{x}{y} = -\text{ctn}\, \theta,$$

and

$$\begin{aligned} \csc (90° + \theta) &= \sec \theta, \\ \sec (90° + \theta) &= -\csc \theta, \\ \text{ctn}\, (90° + \theta) &= -\tan \theta. \end{aligned}$$

These relations can be used to find the trigonometric functions of angles in the second quadrant.

Example 1. Find the value of $\cos 167° 29'.2$.

$$\begin{aligned} \cos 167° 29'.2 &= \cos (90° + 77° 29'.2) \\ &= -\sin 77° 29'.2 \\ &= -0.97624. \end{aligned}$$

EXERCISES

From a figure, derive expressions in terms of functions of θ, for the sine, cosine, and tangent of $90° - \theta$, when θ is defined as below:

1. $180° < \theta < 270°$. **2.** $270° < \theta < 360°$.

From a figure, derive expressions in terms of functions of θ, for the sine, cosine, and tangent of $90° + \theta$ when θ is defined as below:

3. $90° < \theta < 180°$. **4.** $180° < \theta < 270°$.

Derive the relations of Art. 21 using OP, OM, MP, OP_1, etc. as directed lines and taking θ as indicated in each problem:

5. θ in the third quadrant.

6. θ in the fourth quadrant.

Derive the relations of Art. 22 using OP, OM, MP, OP_1, etc. as the directed lines and taking θ as indicated in each problem:

7. θ in the third quadrant.

8. θ in the fourth quadrant.

Use tables and the relations of Arts. 20 and 22 to find the sine, cosine, tangent, and cotangent of the following angles:

9. $167° 29'.8$.	**12.** $-99° 59'.6$.	**15.** $\frac{7\pi}{8}$.
10. $119° 52'.8$.	**13.** $-172° 24'.8$.	**16.** -3.00.
11. $132° 23' 17''$.	**14.** 2.14.	**17.** $-\frac{7\pi}{8}$.

Simplify, expressing results as functions of B:

18. $\sin(90° - B) \cdot \cos(90° + B) \cdot \tan(90° + B) + \cos^2(90° - B)$.

19. $\dfrac{\sin(90° - B)}{\cos(90° + B)} \cdot \dfrac{\sec(90° + B)}{\csc(90° - B)} + \tan(90° + B) \cdot \text{ctn}(90° + B)$.

20. Given $\tan\theta = -\frac{5}{12}$, $90° < \theta < 180°$; find (a) $\sin\left(\frac{\pi}{2} + \theta\right)$; (b) $\sec(90° - \theta)$; (c) $\cos(90° - \theta)$; $\text{ctn}\left(\frac{\pi}{2} + \theta\right)$.

23. Functions of $180° - \theta$ in terms of functions of θ. Let XOP be any angle θ. To illustrate the method, angles in the first and third quadrants have been chosen. Let the

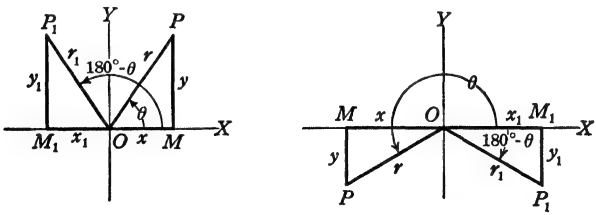


FIG. 33

corresponding angle, $180° - \theta$, be constructed and represented by XOP_1. By taking $OP_1 = OP$, the triangles OMP and OM_1P_1 are made congruent.

For each figure

$$r_1 = r, \quad x_1 = -x, \quad y_1 = y.$$

In each quadrant

$$\sin(180° - \theta) = \frac{y_1}{r_1} = \frac{y}{r} = \sin\theta,$$
$$\cos(180° - \theta) = \frac{x_1}{r_1} = \frac{-x}{r} = -\frac{x}{r} = -\cos\theta,$$
$$\tan(180° - \theta) = \frac{y_1}{x_1} = \frac{y}{-x} = -\frac{y}{x} = -\tan\theta,$$

and

$$\csc(180° - \theta) = \csc\theta,$$
$$\sec(180° - \theta) = -\sec\theta,$$
$$\operatorname{ctn}(180° - \theta) = -\operatorname{ctn}\theta.$$

The relations of this section are those commonly used to find the trigonometric functions of obtuse angles; in this connection they may be written $\sin\theta = \sin(180° - \theta)$, $\cos\theta = -\cos(180° - \theta)$.

Example 1. Find the sine and cosine of 129° 54′.4.

Using the relations above:

$\sin 129° 54'.4 = \sin(180° - 129° 54'.4) = \sin 50° 5'.6 = 0.76709;$
$\cos 129° 54'.4 = -\cos(180° - 129° 54'.4) = -\cos 50° 5'.6 = -0.64154.$

EXERCISES

Find the value of the sine, cosine, and tangent of each of the following angles:

1. 118° 29′.4. **3.** 164° 33′ 12″. **5.** 159° 20′.4.
2. 95° 10′.3. **4.** 172° 48′ 5″. **6.** 141° 0′.6.

7. Find the value of $2\sin\frac{x}{3} - 3\cos\frac{x}{2}$ when $x = 219° 16'.8$.

8. Find the value of $\cos x - 2\cos\frac{x}{2}$ when $x = 124° 6'.2$.

24. Functions of $n \cdot 90° \pm \theta$ in terms of functions of θ. The methods used in the preceding articles of this chapter

may be applied to any integral value of n. As a particular case consider the following example:

Example 1. From a figure derive expressions in terms of functions of θ, for the sine, cosine, and tangent of $270° + \theta$ where $90° < \theta < 180°$.

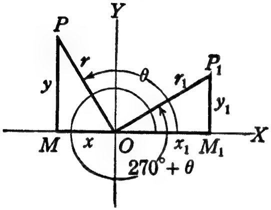


FIG. 34

Let XOP be the given angle θ, where $90° < \theta < 180°$. Let the corresponding angle, $270° + \theta$, be constructed and represented by XOP_1 and let $OP_1 = OP$.

From the congruent triangles OMP and OM_1P_1

$$r_1 = r, \quad y_1 = -x, \quad x_1 = y.$$

Then

$$\sin (270° + \theta) = \frac{y_1}{r_1} = \frac{-x}{r} = -\frac{x}{r} = -\cos \theta,$$

$$\cos (270° + \theta) = \frac{x_1}{r_1} = \frac{y}{r} = \sin \theta,$$

$$\tan (270° + \theta) = \frac{y_1}{x_1} = \frac{-x}{y} = -\frac{x}{y} = -\text{ctn}\, \theta.$$

These equations are true for any other value of θ. In the exercises following, the student will be asked to prove these same relations for other values of θ.

EXERCISES

From a figure, derive expressions, in terms of functions of θ, for the sine, cosine, and tangent of the angles below:

1. $90° + \theta$ where $90° < \theta < 180°$.
2. $90° + \theta$ where $180° < \theta < 270°$.
3. $180° + \theta$ where $90° < \theta < 180°$.
4. $270° - \theta$ where $180° < \theta < 270°$.
5. $270° - \theta$ where $90° < \theta < 180°$.
6. $\theta - 90°$ where $180° < \theta < 270°$.
7. $\frac{3\pi}{2} + \theta$ where $\pi < \theta < \frac{3\pi}{2}$.

8. $360° + \theta$ where $180° < \theta < 270°$.
9. $\theta - 270°$ where $90° < \theta < 180°$.
10. $-270° + \theta$ where $270° < \theta < 360°$.
11. $-180° - \theta$ where $270° < \theta < 360°$.
12. $\frac{3\pi}{2} - \theta$ where $\frac{3\pi}{2} < \theta < 2\pi$.
13. $-360° - \theta$ where $90° < \theta < 180°$.
14. $2\pi - \theta$ where $\frac{\pi}{2} < \theta < \pi$.
15. $450° + \theta$ where $90° < \theta < 180°$.
16. $450° - \theta$ where $180° < \theta < 270°$.

25. Generalization. Whenever trigonometric functions of $n \cdot 90° \pm \theta$ are to be expressed as functions of θ, it is desirable to have a rule for the simplification in place of deriving each relation separately from a figure. In the equations of Arts. 20 to 23, it will be seen that any given function of $n \cdot 90° \pm \theta$ always simplifies to either the same function of θ or to its co-function. It has been emphasized that the equations in those articles are identities. Since the equations are true for all values of θ, the simplest possible case, θ an acute angle, can be used to find the sign of the result. A complete investigation would show the following rules to be true:

When n is an even integer, any function of $n \cdot 90° \pm \theta$ *is numerically equal to the same function of* θ*; when n is an odd integer, any function of* $n \cdot 90° \pm \theta$ *is numerically equal to the co-function of* θ.

The algebraic sign of the result is that of the given function of $n \cdot 90° \pm$ *an acute angle.*

In applying these rules it is advisable to determine the sign first. An error frequently arises from failure to recognize that the required sign is that of the *given* function of $n \cdot 90° \pm \theta$ and not that of the derived function unless they are of the same sign. The examples below illustrate the rules.

Example 1. Express $\tan(270° + \theta)$ as a function of θ.

$270° +$ an acute angle gives an angle of the fourth quadrant, and the tangent of a fourth quadrant angle is negative. Since $270°$ is an odd multiple of $90°$, the co-function of the tangent is obtained.

Therefore $\tan(270° + \theta) = -\operatorname{ctn}\theta.$

Example 2. Express $\cos(-540° - \theta)$ as a function of θ.

$-540° -$ an acute angle gives an angle of the second quadrant, and the cosine of a second quadrant angle is negative. $-540°$ is an even multiple of $90°$, hence the same function, namely the cosine, is obtained.

Therefore $\cos(-540° - \theta) = -\cos\theta.$

Example 3. Express $\sin 289°$ as a function of an acute angle.

$289°$ lies in the fourth quadrant and may be written either as $360° - 71°$ or $270° + 19°$. The relations that apply are:

$$\sin(360° - \theta) = -\sin\theta, \quad \sin(270° + \theta) = -\cos\theta.$$

Then $\sin 289° = \sin(360° - 71°) = -\sin 71°,$

or $\sin 289° = \sin(270° + 19°) = -\cos 19°.$

The relation $\sin(360° - \theta) = -\sin\theta$, may be applied directly; thus

$$\sin 289° = -\sin(360° - 289°) = -\sin 71°.$$

Example 4. Express $\cos 2.18$ as a function of an acute angle and as a decimal.

2.18 is an angle of the second quadrant, hence $\pi - 2.18$ is an acute angle. From Art. 23, $\cos(\pi - \theta) = -\cos\theta$. Hence

$$\begin{aligned} \cos 2.18 &= -\cos(\pi - 2.18) \\ &= -\cos(3.14 - 2.18) \text{ approximately} \\ &= -\cos 0.96 \\ &= -0.574, \text{ from Table I.} \end{aligned}$$

As the approximate value used for π is correct only to three significant figures, the value of this function will not be reliable beyond three significant figures, and possibly only to two. Whenever a function is changing rapidly, for example, the tangent when the angle is near $90°$, the student should especially be on guard against claiming an accuracy that is not warranted.

EXERCISES

Simplify by rule:

1. $\sin(180°+\theta)$.
2. $\cos(90°+\beta)$.
3. $\sin(90°+\alpha)$.
4. $\sin(270°-\alpha)$.
5. $\cos(180°-\theta)$.
6. $\tan(-180°-B)$.
7. $\cos(180°+B)$.
8. $\cos(-270°+\beta)$.
9. $\tan(270°-\theta)$.
10. $\sin(-180°-\theta)$.
11. $\operatorname{ctn}(270°+\phi)$.
12. $\sin(270°+\alpha)$.
13. $\cos(270°-\theta)$.
14. $\tan(-180°+\theta)$.
15. $\sin(540°+\beta)$.
16. $\sin(-360°-\theta)$.
17. $\cos(360°-\theta)$.
18. $\operatorname{ctn}(360°-\alpha)$.
19. $\csc(270°+\beta)$.
20. $\sec(180°-\theta)$.
21. $\sec(360°-A)$.

Simplify: (As each expression is reduced to a function of θ its sign should be shown.)

22. $\cos(270° + \theta) \cdot \sin(180° + \theta) + \sin(90° + \theta) \cdot \sin(270° - \theta)$.
23. $\dfrac{\sin^2(270° - \theta)}{\operatorname{ctn}^2(180° - \theta)} - \dfrac{\sec^2(-\theta) \cdot \cos(180° + \theta)}{\csc^2(90° + \theta)}$.
24. $\sec(\pi - \theta) \cdot \csc\left(\dfrac{\pi}{2} - \theta\right) - \sin\left(\dfrac{3\pi}{2} + \theta\right) \cdot \sec(\pi + \theta)$.
25. $\csc(180° - \theta) \cdot \sec(90° + \theta) + \tan(270° + \theta) \cdot \operatorname{ctn}(180° - \theta)$.

Use tables and rules above to find the sine, cosine, tangent, and cotangent of the following angles:

26. $217° \, 19'.3$.
27. $171° \, 14'.7$.
28. 1.62.
29. $318° \, 22' \, 12''$.
30. $221° \, 52' \, 50''$.
31. 1.89.
32. $-99° \, 9'.8$.
33. $-252° \, 41' \, 16''$.
34. -2.20.

26. Trigonometric functions as directed lines. So far the trigonometric functions have been studied as ratios; but each function can also be represented, both as to magnitude and direction, by a directed line. The method of deriving these will be shown by examples.

Example 1. Derive line values for the sine, cosine, tangent, and secant of an angle in the first quadrant.

Using the ratio definition and a point P on a **unit circle,** that is a circle with a unit radius, $\sin\theta$ equals a fraction, $\dfrac{MP}{OP}$, whose denominator is of unit length, hence the directed line in the numerator must equal $\sin\theta$. The problem of finding a line value for a function

of θ, is one of finding for that function a ratio of directed lines where the line in the denominator is of unit length. This is also illustrated in the case of $\tan \theta$; $\tan \theta = \dfrac{MP}{OM}$ but the denominator is not of unit length. If a line M_1P_1 be drawn as shown in Fig. 35, $\dfrac{MP}{OM} = \dfrac{M_1P_1}{OM_1}$, and $\tan \theta = \dfrac{M_1P_1}{OM_1}$ where the denominator is now a directed line of unit length. Then $\tan \theta = M_1P_1$. Line values for the cosine and secant may be derived in like manner and the results summarized as follows:

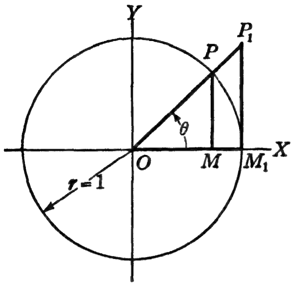


FIG. 35

$$\sin \theta = \frac{MP}{OP} = \frac{MP}{1} = MP,$$

$$\cos \theta = \frac{OM}{OP} = \frac{OM}{1} = OM,$$

$$\tan \theta = \frac{MP}{OM} = \frac{M_1P_1}{OM_1} = \frac{M_1P_1}{1} = M_1P_1,$$

$$\sec \theta = \frac{OP}{OM} = \frac{OP_1}{OM_1} = \frac{OP_1}{1} = OP_1.$$

Example 2. Derive line values for the sine, cosine, tangent, and secant of an angle in the second quadrant.

The general method of derivation has been described in detail in Example 1. This problem shows how to use a directed unit line in the denominator if that line extends in the negative direction. The student should note that the resulting line is in each case of the same sign as the function was found to be in Art. 6. Thus $\tan \theta$ is negative in the second

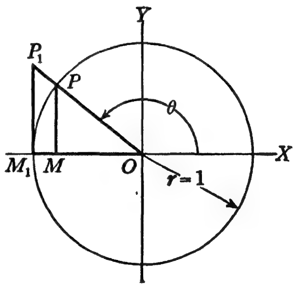


FIG. 36

quadrant, and its line value extends in a negative direction. The solution is shown below.

$$\sin\theta = \frac{MP}{OP} = \frac{MP}{1} = MP,$$

$$\cos\theta = \frac{OM}{OP} = \frac{OM}{1} = OM,$$

$$\tan\theta = \frac{MP}{OM} = \frac{M_1P_1}{OM_1} = \frac{M_1P_1}{-1} = P_1M_1,$$

$$\sec\theta = \frac{OP}{OM} = \frac{OP_1}{OM} = \frac{OP_1}{-1} = P_1O.$$

Example 3. Derive line values for the cotangent and cosecant of an angle in the second quadrant.

Applying the general method of the preceding examples, $\csc\theta = \frac{OP}{MP}$. As MP is not of unit length it will be necessary to find another ratio that is equal to $\frac{OP}{MP}$ where the denominator is of unit length. A point, P', on the terminal side of θ is found as shown in Fig. 37. Then

$$\frac{OP}{MP} = \frac{OP'}{M'P'} = \frac{OP'}{1} = OP',$$

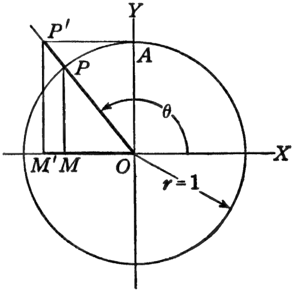


FIG. 37

and $\csc\theta = OP'$. A line value for the cotangent can be found from the same figure, and the results summarized:

$$\csc\theta = \frac{OP}{MP} = \frac{OP'}{M'P'} = \frac{OP'}{1} = OP',$$

$$\text{ctn}\,\theta = \frac{OM}{MP} = \frac{OM'}{M'P'} = \frac{OM'}{1} = OM'.$$

The line values for the trigonometric functions, derived as shown above, are summarized below for angles in the four quadrants and the lines are shown in Fig. 38.

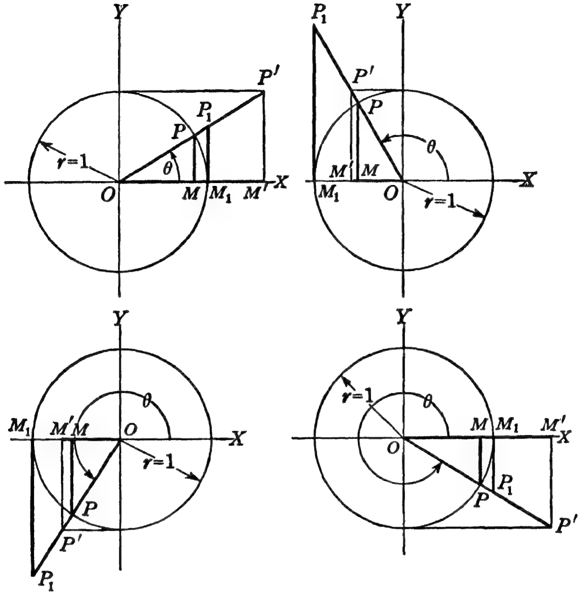


FIG. 38

Angle	θ_1	θ_2	θ_3	θ_4
sine	MP	MP	MP	MP
cosine	OM	OM	OM	OM
tangent	M_1P_1	P_1M_1	P_1M_1	M_1P_1
secant	OP_1	P_1O	P_1O	OP_1
cosecant	OP'	OP'	$P'O$	$P'O$
cotangent	OM'	OM'	$M'O$	$M'O$

EXERCISES

Derive line values for the sine, cosine, tangent, and secant of the following angles:

1. θ_3. **2.** θ_4. **3.** $450° < \theta < 540°$.

Derive line values for the cosecant and cotangent of the following angles:

4. θ_1. **5.** θ_3. **6.** θ_4.

Verify the line values as given in this article for the following angles:

7. θ_1. **8.** θ_3. **9.** θ_4.

10. Show that the sign of the line value definitions of each trigonometric function of a second quadrant angle is the same as the sign given by the ratio definition.

11. Prove formulas [5] and [6] by using line values and taking the angle in the second quadrant.

12. Given an angle θ in the second quadrant, derive line values for the sine and cosine of θ and $90° + \theta$. From these prove

$$\sin(90° + \theta) = \cos\theta \quad \text{and} \quad \cos(90° + \theta) = -\sin\theta$$

for this value of θ.

27. Functions of 0°, 90°, 180°, and 270°. The trigonometric functions of the quadrantal angles may be defined through line values. To find these functions, the quadrantal angles are considered as the limits of angles near them and approaching them.

Referring to Fig. 39, the functions of 0° may be defined:

$$\sin 0° = \lim_{\theta\to 0°} \sin\theta = \lim_{\theta\to 0°} MP = 0,$$
$$\cos 0° = \lim_{\theta\to 0°} \cos\theta = \lim_{\theta\to 0°} OM = 1,$$
$$\tan 0° = \lim_{\theta\to 0°} \tan\theta = \lim_{\theta\to 0°} M_1P_1 = 0.$$

From Fig. 39, the functions of 90° may be defined:

$$\sin 90° = \lim_{\theta\to 90°} \sin\theta = \lim_{\theta\to 90°} MP = 1,$$
$$\cos 90° = \lim_{\theta\to 90°} \cos\theta = \lim_{\theta\to 90°} OM = 0,$$
$$\tan 90° = \lim_{\theta\to 90°} \tan\theta = \lim_{\theta\to 90°} M_1P_1;$$

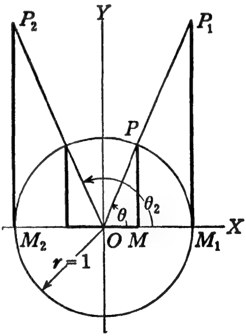


FIG. 39

M_1P_1 is a positive quantity and increases without limit as $\theta \to 90°$. The conventional symbol for a positive quantity that increases without limit is $+\infty$, hence $\tan 90° = +\infty$.

If the angle used to define the functions of 90° had been taken in the second quadrant,

$$\sin 90° = 1 \quad \text{and} \quad \cos 90° = 0$$

as above, though from different lines. With this new choice of angles,

$$\tan 90° = \lim_{\theta_2 \to 90°} \tan \theta_2 = \lim_{\theta_2 \to 90°} P_2M_2;$$

P_2M_2 is a negative quantity and increases in length without limit as $\theta_2 \to 90°$. The conventional symbol for such a quantity is $-\infty$. Tan 90° then is either $+\infty$ or $-\infty$ according as 90° is approached from the first or second quadrant.

From a similar process all the functions of 180° and 270° can be found. The results for all quadrantal angles are indicated in the table below:

Angle	0°	90°	180°	270°
sine	0	1	0	-1
cosine	1	0	-1	0
tangent	0	$+\infty$ or $-\infty$	0	$+\infty$ or $-\infty$

EXERCISES

From a figure, derive the values of the sine, cosine, and tangent, of the following angles:

1. 180°. **2.** 270°. **3.** 360°.

4. Complete the table below:

Angle	0	π	2π	$\frac{\pi}{2}$	$\frac{3\pi}{2}$
sine	0				
cosine	1				
tangent					

Simplify the following:

5. $\dfrac{\sin 90° + \cos 90°}{\sin 180° + \cos 180°}$

7. $\dfrac{\cos 0° + \cos 90° + \cos 180°}{\tan 0° + \tan 90° + \tan 180°}$.

6. $\dfrac{\cos \frac{3\pi}{2} \cdot \cos \pi}{\sin \frac{\pi}{2} + \sin \pi}$.

8. $\dfrac{\sin 0° + \tan 180° + \sin \frac{3\pi}{2}}{\tan 0° + \sin \pi + \cos 0°}$.

28. Variations in the trigonometric functions. The line value definitions give a simple method of tracing the variations in the functions of an angle as the angle varies from 0 to 2π and beyond. In the circle described by P rotating about O, $OP = 1$, the changes in MP represent the variations in the sine. The results are put in tabular form below:

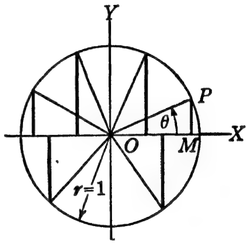


FIG. 40

As the angle varies from	$0 \to \frac{\pi}{2}$	$\frac{\pi}{2} \to \pi$	$\pi \to \frac{3\pi}{2}$	$\frac{3\pi}{2} \to 2\pi$	$2\pi \to \frac{5\pi}{2}$
its sine varies from	$0 \to 1$	$1 \to 0$	$0 \to -1$	$-1 \to 0$	$0 \to 1$

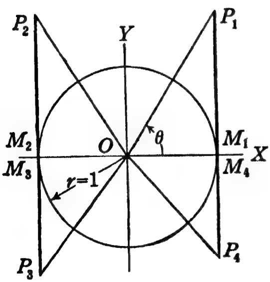


FIG. 41

Likewise in the unit circle of Fig. 41 the changes in M_1P_1 as θ changes in the first quadrant, show the variations of the tangent in that quadrant. In the second and third quadrants, the tangents are read from P to M and in the fourth from M to P, and the variations in these quantities indicate the variations in the tangent of angles in those quadrants. The results are indicated in tabular form below:

As the angle varies from	$0 \to \frac{\pi}{2}$	$\frac{\pi}{2} \to \pi$	$\pi \to \frac{3\pi}{2}$	$\frac{3\pi}{2} \to 2\pi$	$2\pi \to \frac{5\pi}{2}$
its tangent varies from	$0 \to \infty$	$-\infty \to 0$	$0 \to \infty$	$-\infty \to 0$	$0 \to \infty$

EXERCISES

1. Using Fig. 40, determine how $\cos\theta$ varies as θ varies from 0 to 2π.
2. Using Fig. 41, determine how $\sec\theta$ varies as θ varies from 0 to 2π.
3. Complete the outline below:

As θ varies from	$0 \to \frac{\pi}{4}$	$\frac{\pi}{4} \to \frac{\pi}{2}$	$\frac{\pi}{2} \to \frac{3\pi}{4}$	$\frac{3\pi}{4} \to \pi$	$\pi \to \frac{5\pi}{4}$	$\frac{5\pi}{4} \to \frac{3\pi}{2}$
its sine varies from						
its cosine varies from						
its tangent varies from						

29. Periodicity of the trigonometric functions. A study of the variations in MP of Fig. 40 will show that as the angle varies from 0 to 2π, the sine varies through all of its possible values and returns at 2π to the value it had at 0. Further study will show that as the angle varies from 2π to 4π, the values of the sine repeat the values taken as the angle varies from 0 to 2π, and that these values are repeated again for each 2π. For this reason the sine is called a **periodic function** of 2π. The repeating character of the sine function is shown in Fig. 66 where values of the angle are plotted as abscissas and the corresponding values of the sine as ordinates, and the graph shows the geometrical meaning of a period. The tangent varies from $-\infty$ to $+\infty$ in each period of π starting with any odd multiple of $\frac{\pi}{2}$; therefore the tangent is a periodic function of period π.

The graph of the tangent in Fig. 67 shows its repeating character and the meaning of a period. A method for finding the period of the functions of other angles is shown in the examples below.

Example 1. Find the period of $\sin \frac{2x}{3}$.

The values of $\sin \theta$ begin to repeat after θ has varied from 0 to 2π. Likewise the values of $\frac{2x}{3}$ will begin to repeat when $\frac{2x}{3}$ has varied from 0 to 2π since the angle used may be called θ or $\frac{2x}{3}$.

As $\frac{2x}{3}$ varies from 0 to 2π, x varies from $\frac{3}{2}\cdot 0$ to $\frac{3}{2}\cdot 2\pi$, or from 0 to 3π. Hence the period of $\sin\frac{2x}{3}$ is 3π. The period is shown in the graph of $y = 2\sin\frac{2x}{3}$, Fig. 68.

Example 2. Find the period of $\sin\frac{\pi x}{2}$.

As $\frac{\pi x}{2}$ varies from 0 to 2π, x varies from $\frac{2}{\pi}\cdot 0$ to $\frac{2}{\pi}\cdot 2\pi$, or from 0 to 4. Hence the period of $\sin\frac{\pi x}{2}$ is 4.

Example 3. Find the period of $\tan(3x + 1)$.

To give a period for $\tan\theta$, θ varies through π.

As $3x + 1$ varies from 0 to π, $3x$ varies from $0 - 1$ to $\pi - 1$, and x varies from $-\frac{1}{3}$ to $\frac{\pi}{3} - \frac{1}{3}$. Hence the period of $\tan(3x + 1)$ is $\left(\frac{\pi}{3} - \frac{1}{3}\right) - \left(-\frac{1}{3}\right)$, or $\frac{\pi}{3}$.

EXERCISES

Find the period of the following:

1. $\cos\theta$.	**3.** $\cos\frac{2\pi x}{3}$.	**5.** $\cos 2x$.
2. $\sec\theta$.	**4.** $\sin\frac{x}{3}$.	**6.** $\tan\frac{\pi x}{4}$.

7. $\tan 2x$.

8. $\sin \frac{6\pi x}{7}$.

9. $\cos \pi x$.

10. $\operatorname{ctn} \frac{4x}{3}$.

11. $\csc \frac{2x}{3}$.

12. $\sin\left(4x - \frac{\pi}{2}\right)$.

13. $\cos(3x + 2)$.

14. $\operatorname{ctn}\left(\frac{2x+\pi}{4}\right)$.

15. $\csc\left(x - \frac{\pi}{2}\right)$.

Compare the periods of the following:

16. $\sin(4x-3)$ and $\sin 4x$.

17. $\cos(x-\pi)$ and $\cos(2x-\pi)$.

18. $\tan(ax)$ and $\tan(ax+b)$.

19. $\cos(bx+d)$ and $\cos(bx+c)$.

For what value of a will sin ax have the periods indicated below:

20. π.

21. 2π.

22. $\frac{3\pi}{2}$.

23. 2.

30. Inverse trigonometric functions. In the equation $u = \sin v$, v is an angle whose sine is u. This is expressed in mathematical symbols by $\boldsymbol{v} = \textbf{arc sin}\ \boldsymbol{u}$, or by $\boldsymbol{v} = \boldsymbol{\sin^{-1} u}$. If the latter symbol is used, it is to be understood that the -1 is a part of the symbol and not an exponent, hence $\frac{1}{\sin\theta}$ must be written $(\sin\theta)^{-1}$. As the only number used in this symbol is 1, numbers other than 1 may be used for exponents as in algebra; thus $\cos^{-2}\theta = \frac{1}{\cos^2\theta} = (\cos\theta)^{-2}$. In the exercises following, the symbols arc sin u and $\sin^{-1} u$ will be used interchangeably to acquaint students with both notations.

This new symbol, in either form, is read as **inverse sine u**, as **arc sin u**, or better yet, as **an angle whose sine is *u*.** The latter emphasizes that arc sin u is an angle. Arc cos u is defined from the equation $u = \cos v$ in a similar manner and is also an angle. Likewise there may be defined arc tan u, and other inverse functions to correspond to the trigonometric functions, giving a group as follows:

arc sin u,	**arc tan u,**	**arc sec u,**
arc cos u,	**arc ctn u,**	**arc csc u.**

These six quantities are angles.

The trigonometric functions are single valued; thus if A is given, $\sin A$ has only one value. On the contrary, the inverse functions are multiple valued; for example since $\sin 30° = \frac{1}{2}$ and $\sin 150° = \frac{1}{2}$, arc sin $\frac{1}{2} = 30°, 150°$, or any angle coterminal with them. Likewise arc cos $\frac{\sqrt{3}}{2} = 30°$, $330°$, or any angle coterminal with them. If arc tan $\frac{12}{5}$ is to be limited to the angle in the third quadrant, a convenient notation is arc $\tan_3 \frac{12}{5}$, or if the angle is to be called A, by the notation $A_3 = \text{arc tan } \frac{12}{5}$. The meaning of the inverse functions is also shown by the examples following.

Example 1. Find the angles from 0° to 360° represented by arc sin $(-\frac{1}{2})$.

Let $A = \text{arc sin } (-\frac{1}{2})$, then $\sin A = -\frac{1}{2}$ and A must lie in the third or fourth quadrants. The acute angle whose sine is $\frac{1}{2}$ is 30°. Hence

$$A = 180° + 30° = 210°, \quad \text{or} \quad A = 360° - 30° = 330°, \quad \text{and}$$
$$\text{arc sin } (-\tfrac{1}{2}) = 210° \text{ or } 330°.$$

Example 2. What positive angles less than 360° are represented by $\cos^{-1} 0.57624$?

Let $A = \cos^{-1} 0.57624$, then $\cos A = 0.57624$, and A lies in the first or fourth quadrants. From tables, the acute angle whose cosine is 0.57624 is found to be 54° 48′.8. Hence the angle is 54° 48′.8 or 360° − 54° 48′.8 and

$$\cos^{-1} 0.57624 = 54° \, 48'.8 \quad \text{or} \quad 305° \, 11'.2.$$

Example 3. Simplify cos [arc tan $(-\frac{3}{4})$].

Let $A = \text{arc tan } (-\frac{3}{4})$. Then $\tan A = -\frac{3}{4}$ and there is an angle A in the second quadrant and another in the fourth. The two angles are shown in the figure below as A_2 and A_4.

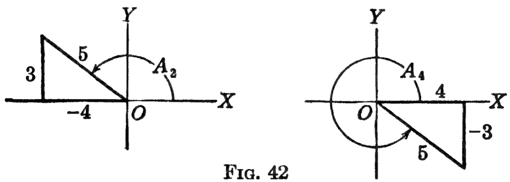


FIG. 42

The problem now becomes one of finding $\cos A$ and from Fig. 42

$$\cos A_2 = -\tfrac{4}{5}, \qquad \cos A_4 = \tfrac{4}{5}.$$

Hence

$$\cos [\text{arc}\tan_2 (-\tfrac{3}{4})] = -\tfrac{4}{5}, \qquad \cos [\text{arc}\tan_4 (-\tfrac{3}{4})] = \tfrac{4}{5},$$

and

$$\cos [\text{arc}\tan (-\tfrac{3}{4})] = -\tfrac{4}{5} \quad \text{or} \quad +\tfrac{4}{5}.$$

EXERCISES

Find all angles from 0° to 360° that are represented by the following expressions:

1. $\sin^{-1} \frac{\sqrt{3}}{2}$.

2. $\text{arc}\tan (-\sqrt{3})$.

3. $\text{arc}\sin (-\frac{1}{2})$.

4. $\text{arc}\sin (-1)$.

5. $\cos^{-1} \frac{1}{2}$.

6. $\text{arc}\sin (-0.78691)$.

7. $\text{arc ctn} (-0.78652)$.

8. $\text{arc}\cos (-0.01986)$.

9. $\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$.

10. $\cos^{-1} 0.98762$.

11. $\tan^{-1} 0.07954$.

12. $\text{arc}\tan (-1.5784)$.

Express in π radians all positive angles from 0 to 2π, inclusive, that are represented by the following expressions:

13. $\text{arc}\tan 1$.

14. $\text{arc ctn} (-1)$.

15. $\text{arc}\cos\left(-\frac{\sqrt{3}}{2}\right)$.

16. $\text{arc}\sin 0$.

17. $\text{arc}\cos (-1)$.

18. $\text{arc}\sin\left(-\frac{\sqrt{2}}{2}\right)$.

19. $\text{arc}\tan (-\infty)$.

20. $\text{arc ctn}\, 0$.

21. $\text{arc}\cos 0$.

Simplify the following:

22. $\sin (\sin^{-1} a)$.

23. $\sin^{-1}\left(\sin \frac{\pi}{3}\right)$.

24. $\tan [\text{arc}\tan (-1)]$.

25. $\text{arc}\tan \left[\tan \frac{3\pi}{4}\right]$.

26. $\sin^{-1} (\cos 35°)$.

27. $\sin (\text{arc}\tan_3 \frac{5}{12})$.

Express each of the following by two inverse functions other than the one given, using only the acute angle in each case:

28. $\text{arc}\tan \frac{\sqrt{1 - x^2}}{x}$.

29. $\text{arc}\sin \frac{\sqrt{16 - 9x^4}}{4}$.

30. $\text{arc}\tan \frac{\sqrt{x^2 + 4}}{\sqrt{5 - 6x}}$.

31. $\cos^{-1} \frac{e^x}{\sqrt{e^{2x} + e^{-2x}}}$.

31. Simple trigonometric equations. In algebra, equations are divided into two kinds, identical and conditional.

The same classification applies to equations involving trigonometric functions. The equations hitherto mentioned have been identical equations, though more commonly called identities. Conditional equations involving trigonometric functions are usually known as **trigonometric equations**. The solutions of some simple trigonometric equations are shown in the examples below.

Example 1. Find the angles from 0° to 360° that satisfy $(2 \sin x - 1)(\cos x - 2)(\tan x + \frac{3}{2}) = 0$.

This equation is satisfied when either

$2 \sin x - 1 = 0$, or	$\cos x - 2 = 0$, or	$\tan x + \frac{3}{2} = 0$.
$\sin x = \frac{1}{2}$ There is a value for x in the first quadrant and another in the second. $x = 30°, 150°$.	$\cos x = 2$ No value of x will satisfy this equation.	$\tan x = -\frac{3}{2}$ There is a value for x in the second quadrant and another in the fourth. From tables, arc tan 1.5000 = 56°18′.6. $x = 123°41'.4, 303°41'.4$.

Example 2. Solve $\sin^2 x - \cos^2 x - \sin x = 0$, for $0° < x < 360°$.

It frequently is an advantage to change to one function of the unknown angle.

Changing to sines, $\sin^2 x - 1 + \sin^2 x - \sin x = 0$.
Simplifying, $2 \sin^2 x - \sin x - 1 = 0$.
Factoring, $(2 \sin x + 1)(\sin x - 1) = 0$.

The given equation is satisfied when either

$2 \sin x + 1 = 0$, or	$\sin x - 1 = 0$.
$\sin x = -\frac{1}{2}$ There is a value for x in the third quadrant and another in the fourth. $x = 210°, 330°$.	$\sin x = 1$ $x = 90°$.

Example 3. Solve $3\cos^2\theta - 2\sin^2\theta + 6\cos\theta = 0$, for $0° < \theta < 360°$.

Changing to cosines, $3\cos^2\theta - 2(1 - \cos^2\theta) + 6\cos\theta = 0.$

Simplifying, $5\cos^2\theta + 6\cos\theta - 2 = 0.$

This equation cannot be solved by factoring, but the equation is a quadratic in $\cos\theta$. Hence, applying the quadratic formula,

$$\cos\theta = \frac{-6 \pm \sqrt{76}}{10} = -1.47178 \quad \text{or} \quad 0.27178.$$

The given equation is satisfied when either

$\cos\theta = -1.47178$, or	$\cos\theta = 0.27178$.
There is no value of θ that satisfies this equation.	From tables. $\cos^{-1} 0.27178 = 74° 13'.8$. $\theta = 74° 13'.8$ or $285° 46'.2$.

EXERCISES

Find the angles from 0° *to* 360° *that satisfy the following equations:*

1. $\sin\theta = -1.$
2. $\sin\theta = -\frac{\sqrt{2}}{2}.$
3. $\cos\theta = -1.$
4. $\tan\alpha = 1.$
5. $\cos\theta = 0.$
6. $\tan\theta = -\frac{\sqrt{3}}{3}.$
7. $\sin\theta = -0.07817.$
8. $\cos x = -0.78543.$
9. $\sin x = -0.89718.$
10. $\tan\beta = \frac{3}{2\sqrt{5}}.$
11. $\cos x = 0.01521.$
12. $\tan\theta = -2.5700.$
13. $\cos x + 8 = 12 - 3\cos x.$
14. $(\sin\theta - 3)(2\sin\theta + 1) = 0.$
15. $\sin x + \cos x = 0.$
16. $\sin\theta\cos\theta = 0.$
17. $(\cos\theta - 1)(2\cos\theta - 1) = 0.$
18. $(\tan\beta + 1)(\tan\beta + \sqrt{3}) = 0.$
19. $2\sin\theta(2\cos\theta + \sqrt{2}) = 0.$
20. $\sin\theta\cos\theta = \sin^2\theta.$
21. $2\sin^2 x + \sin x - 3 = 0.$
22. $(\cos\theta - 2)(\sin x + 4) = 0.$
23. $4\sin^2\phi - 17\sin\phi + 4 = 0.$
24. $4\sec^2\theta = 17\tan\theta.$
25. $4\cos^4 x = \cos^2 x.$
26. $\tan^2 x + \sec^2 x + \tan x = 4.$
27. $2\cos^2\theta - 5\sin\theta - 3 = 0.$
28. $\cos^2\theta + 2 = 4\cos\theta.$
29. $3\sin^2\theta + \cos\theta + 3 = 0.$
30. $2\sin^2\theta + 4\sin\theta - 7 = 0.$
31. $\tan^2 A - 2\tan A - 1 = 0.$
32. $3\tan^2\beta + 5\sec\beta + 4 = 0.$
33. $\sin\theta(2\sin\theta - 1)(2\cos\theta - 1) = 0.$
34. $3\sin^2\alpha - 3\cos^2\alpha - 7\cos\alpha + 5 = 0.$

GENERAL EXERCISES

1. Given $\cos\theta = \frac{5}{13}$ and $180° < \theta < 360°$; find the functions of $(-\theta)$.

2. Given $\cos(-\theta) = \frac{5}{13}$ and $180° < \theta < 360°$; find the functions of θ.

3. Simplify: $\dfrac{\sin(180° + x)}{\cos(270° + x)} + \dfrac{\tan(90° + x)\cdot\sin(270° + x)}{\sec(90° + x)}$.

4. Given $\tan(-\theta) = -\frac{8}{15}$ and $180° < \theta < 360°$; find the value of $\cos(180° - \theta)\csc(90° - \theta) + \tan(270° - \theta)\,\text{ctn}\,\theta$.

5. Simplify: $\sin^2(-C) - \sin^2 C + \tan(-C)\cos(-C) + \sin(-C)$.

6. From a figure, derive expressions in terms of functions of θ for the sine, cosine, and tangent of $270° + \theta$ where $90° < \theta < 180°$.

7. Express by means of a figure, the sine, cosine, and tangent of $270° + \theta$ where $270° < \theta < 360°$.

8. Simplify, using the smallest positive value for each inverse function:

$$\sin 90° + \sin[\text{arc}\cos(-1)] + \cos[\text{arc}\sin(-1)].$$

9. If $\theta_3 = \text{arc}\tan\dfrac{e^x - e^{-x}}{2}$, find the trigonometric functions of θ_3.

10. Show by using line values, that the cosine of 180° is the same whether 180° is considered the limit of a second or a third quadrant angle.

11. Derive a line value for the tangent of an angle in the fourth quadrant, and use this line value to find tan 270°.

12. Evaluate: $\dfrac{\cos 90° + \sin 90°\cdot\cos 180°}{\tan 0°\cdot\cos 210° + \sin 270°}$.

13. Evaluate: $\dfrac{\cos(-30°) + \sin 0° + \tan 180°}{\sin 270° + \cos 150°}$.

14. How many values has

$$\cos 180° + \sin[\cos^{-1} 0] + \tan\left[\text{arc}\cos\left(-\frac{\sqrt{2}}{2}\right)\right]$$

if each inverse function represents only positive angles less than 360°? What are these values?

15. Express the sine, cosine, and tangent of $(-270° - \theta)$ where $90° < \theta < 180°$, in terms of functions of θ, by means of a figure.

16. From a figure, derive expressions in terms of functions of θ, for the sine, cosine, and tangent of $\frac{\pi}{2} + \theta$ where $\pi < \theta < \frac{3\pi}{2}$.

17. Determine if it is possible for $\cos(270° + \theta)$ to equal $\cos\theta$. If it is possible, determine whether $\cos(270° + \theta) = \cos\theta$ is an identity or a conditional equation.

18. Determine if $\cos(270° + \theta) = -\sin\theta$ is an identity or a conditional equation.

19. For what values of θ is $\sin(180° + \theta) = \cos(270° - \theta)$?

20. By use of line values, prove formulas [5] and [6] for angles of the third quadrant.

21. Prove formulas [5] and [7] for angles of the second quadrant by use of line values.

22. Evaluate to five significant figures:

$$(\cos 98°\,8'\,12'' + \tan 200°\,19'\,25'' + \sin 299°\,49'\,58'')^2.$$

23. Evaluate to five significant figures:

$$\sqrt[3]{\frac{\sin 348°\,16'.9}{\cos 216°\,46'.3 + \operatorname{ctn} 90°}}.$$

24. If $x_2 = \operatorname{arc}\cos\left(-\frac{8}{17}\right)$, evaluate:

$$\frac{\sin(180° - x)}{\sin(270° + x)} \cdot \frac{\sin(-x)}{\sin(90° + x)} + \frac{\sin(270° - x)}{\cos(-x)}.$$

25. If $x = 232°\,7'.8$, evaluate:

$$\frac{\sin(\pi - x)}{\sec\left(\frac{\pi}{2} + x\right)} - \frac{\sin\left(\frac{3\pi}{2} + x\right)}{\sec(-x)}.$$

26. Evaluate: $$\frac{\tan\frac{3\pi}{2} + \sin\left(-\frac{5\pi}{6}\right)}{\sin\frac{\pi}{2} + \cos\pi - \sin\frac{3\pi}{2}}.$$

27. Simplify: $$\frac{\tan(-\pi)\cdot\cos\frac{2\pi}{3}}{\tan\frac{\pi}{2}\cdot\cos 0\cdot\sin\frac{\pi}{2}}.$$

28. By using line values with suitable figures, prove $\sin(180° - \theta) = \sin\theta$, where $90° < \theta < 180°$.

29. If θ is an angle of the third quadrant, derive line values for $\sin\theta$, $\cos\theta$, $\sin(90° + \theta)$, and $\cos(90° + \theta)$. From these values show that the relations

$$\sin(90° + \theta) = \cos\theta \quad \text{and} \quad \cos(90° + \theta) = -\sin\theta.$$

are true for this particular value of θ.

30. Show the difference, if any, between the following:

$$\sin(\text{arc}\sin a) \quad \text{and} \quad \text{arc}\sin(\sin a).$$

31. If $x = 126° 52'.2$, find the value of

$$\text{ctn}\left(\frac{\pi}{2} + x\right) \cdot \frac{\cos\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} + x\right)} + \sec\left(\frac{\pi}{2} + x\right)\cos\left(\frac{\pi}{2} + x\right).$$

32. Determine if the following equation is an identity:

$$\frac{\cos 0°}{1 - \cos(-\theta)} + \frac{\sin 90°}{1 - \cos(180° + \theta)} = 2\csc^2(-\theta).$$

33. Determine if the following equation is an identity:

$$\cos^2\theta - \sin^2(-\theta) = 2\cos^2(-\theta) + \cos 180°.$$

34. By what values of α is the following equation satisfied:

$$\frac{\sin(180° - \alpha)}{1 - \sin(270° + \alpha)} = \frac{1 + \sin(270° - \alpha)}{\sin(180° - \alpha)}.$$

35. Prove: $\dfrac{\tan(180° + \alpha) - \tan(180° - \beta)}{\tan(270° - \alpha) + \text{ctn}(180° + \beta)} = \tan\alpha\tan\beta.$

36. Evaluate, using for each inverse function the smallest positive angle that it represents:

$$\text{arc}\tan(\cos 0°) + \text{arc}\sin(\cos 135°) + \text{arc}\cos(\sin 225°).$$

37. Simplify, using for each inverse function the smallest positive angle greater than $\frac{\pi}{2}$ that it represents:

$$\text{arc}\sin\left(\cos\frac{\pi}{2}\right) + \text{arc}\cos\left(\tan\frac{\pi}{4}\right) + \text{arc}\tan\left(\sin\frac{3\pi}{2}\right).$$

38. If $A_2 = \text{arc}\sin\dfrac{2\sqrt{ab}}{a + b}$ and $a > b$, find $\tan A$.

39. Simplify: $\sin^2(-B) + \cos^2(-B) + \tan^2(-B)$.

40. Evaluate to five significant figures:

$$\left[\frac{\sin 241^\circ\, 31'.8 + \cos 281^\circ\, 43'.7}{\tan 100^\circ\, 10'.1}\right]^{\frac{1}{2}}.$$

41. Evaluate to five significant figures:

$$\sqrt[3]{\frac{\sin 104^\circ\, 19'.3 + \cos 222^\circ\, 9'.2}{\tan 289^\circ\, 46'.4}}.$$

42. Classify the following as identities or conditional equations, giving reasons in each case:

(a) $\cos(270^\circ + \theta) = \cos\theta$;
(b) $\cos(270^\circ + \theta) = \sin\theta$;
(c) $\tan(90^\circ + \theta) = \operatorname{ctn}\theta$;
(d) $\tan(180^\circ - \theta) = \tan(270^\circ - \theta)$;
(e) $\cos(180^\circ + \theta) = \sin(270^\circ - \theta)$;
(f) $\cos(270^\circ + \theta) = \sin(360^\circ - \theta)$.

43. Prove: $\dfrac{\cos(270^\circ + \alpha)}{1 - \cos(180^\circ - \alpha)} = \dfrac{1 - \cos(-\alpha)}{\cos(90^\circ - \alpha)}$.

44. For what values of α is

$$\sec(270^\circ + \alpha)\csc(-\alpha)\sec(-\alpha)\csc(270^\circ - \alpha) = \csc^2\alpha + \sec^2\alpha?$$

45. Find A, $0^\circ < A < 360^\circ$, if $\cos A = \dfrac{\sin 210^\circ\, 26'.2}{\operatorname{ctn} 325^\circ\, 46'.7}$.

46. Find B, $0^\circ < B < 360^\circ$, if $\operatorname{ctn} B = \dfrac{\cos 171^\circ\, 18'.2}{\tan 301^\circ\, 49'.6}$.

47. Find A, $0^\circ < A < 360^\circ$, if $\sin A = \dfrac{\cos 303^\circ\, 22'.4}{\tan 110^\circ\, 27'.3}$.

48. Find B, $0^\circ < B < 360^\circ$, if $\tan B = \dfrac{\sin 116^\circ\, 28'.5}{\cos 126^\circ\, 31'.6}$.

49. Evaluate to five places: $\dfrac{\csc 138^\circ\, 19'.8 \cdot \operatorname{ctn} 108^\circ\, 30'.5}{\sec 317^\circ\, 18'.5}$.

50. Evaluate to five places:

$$\frac{\log\sin 99^\circ\, 11'.1 + \log\cos 289^\circ\, 49'.2}{\sin 99^\circ\, 11'.1 - \cos 289^\circ\, 49'.2}.$$

CHAPTER IV

RELATIONS BETWEEN THE TRIGONOMETRIC FUNCTIONS OF SEVERAL ANGLES

32. Introduction. The articles of this chapter are devoted mainly to the development of certain standard formulas involving trigonometric functions. These formulas will be useful in many branches of mathematics, physics and allied subjects. In calculus especially, the student will need them many times. The development of these formulas will depend on a theorem concerning the area of a triangle, the proof of which is given in the following article.

33. The area of a triangle in terms of two sides and the included angle. In the triangle ABC of Fig. 43, let each

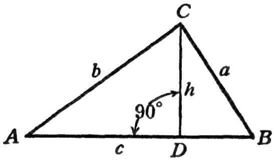


FIG. 43

angle be acute and let a, b, and c be the sides, h the altitude, and K the area.

$$K = \tfrac{1}{2}\,hc.$$

But from the right triangle ADC

$$\frac{h}{b} = \sin A \quad \text{or} \quad h = b \sin A.$$

Hence

$$\mathbf{K = \tfrac{1}{2}\,bc \sin A.} \qquad [10a]$$

If one angle of the triangle is obtuse the same expression for the area of the triangle is obtained. Using the triangle of Fig. 44:

$$K = \tfrac{1}{2}\,hc.$$

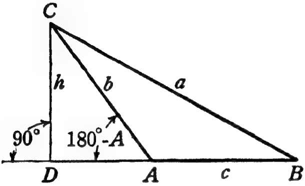


FIG. 44

But from the right triangle ADC

$$\frac{h}{b} = \sin(180^\circ - A)$$
$$= \sin A,$$

or

$$h = b \sin A.$$

Hence

$$\mathbf{K = \tfrac{1}{2}\,bc \sin A} \qquad [10a]$$

and in like manner

$$\mathbf{K = \tfrac{1}{2}\,ac \sin B} \qquad [10b]$$

$$\mathbf{K = \tfrac{1}{2}\,ab \sin C.} \qquad [10c]$$

The two proofs above establish the theorem:

The area of a triangle is equal to one half the product of two sides times the sine of the included angle.

34. The sine of the sum of two angles. Given the angles, α and β, where $\alpha + \beta < 180^\circ$, and let the angles be so placed that their sum is the angle DAB as in Fig. 45. Let DB be drawn perpendicular to the common side of the angles, and the lengths of the sides be denoted by l, m, and n. respectively.

$$K(DAB) = K(DAC) + K(ABC)^*$$

and by the theorem of Art. 33,

$$\tfrac{1}{2}\,ln \sin(\alpha + \beta) = \tfrac{1}{2}\,lm \sin \alpha + \tfrac{1}{2}\,mn \sin \beta.$$

* The symbol $K(ABC)$ indicates the area of the triangle ABC.

Dividing both sides by $\frac{1}{2}\,ln$, and simplifying,

$$\sin(\alpha+\beta) = \frac{m}{n}\sin\alpha + \frac{m}{l}\sin\beta.$$

But from the right triangle ACB

$$\frac{m}{n} = \cos\beta$$

and from the right triangle ACD

$$\frac{m}{l} = \cos\alpha.$$

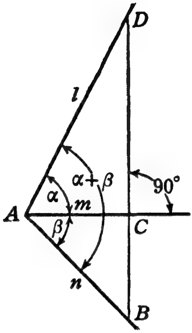


FIG. 45

Hence

$$\mathbf{\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.} \qquad [11]$$

When $180° < \alpha+\beta < 360°$, the same law is true. In Fig. 46

$$K(BAD) = K(BCA) + K(ACD)$$

or

$$\tfrac{1}{2}\,ln\sin[(180°-\alpha)+(180°-\beta)] = \tfrac{1}{2}\,lm\sin(180°-\alpha) + \tfrac{1}{2}\,mn\sin(180°-\beta).$$

Dividing both sides of the equation by $\frac{1}{2}\,ln$ and simplifying,

$$\sin[360° - (\alpha+\beta)] = \frac{m}{n}\sin(180°-\alpha) + \frac{m}{l}\sin(180°-\beta).$$

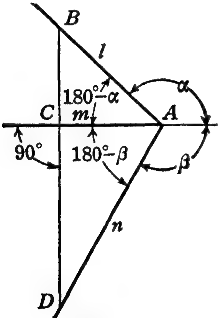


FIG. 46

But

$$\sin[360° - (\alpha+\beta)] = -\sin(\alpha+\beta),$$
$$\sin(180°-\alpha) = \sin\alpha,$$
$$\sin(180°-\beta) = \sin\beta,$$
$$\frac{m}{n} = \cos(180°-\beta) = -\cos\beta,$$

and

$$\frac{m}{l} = \cos(180°-\alpha) = -\cos\alpha.$$

Hence

$$-\sin(\alpha+\beta) = -\sin\alpha\cos\beta - \cos\alpha\sin\beta$$

or

$$\mathbf{\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.} \qquad [11]$$

By a similar proof this law can be shown to be true for all values of α and β, hence the law:

The sine of the sum of two angles equals the sine of the first times the cosine of the second plus the cosine of the first times the sine of the second.

An alternate proof for the formula for sin ($\alpha + \beta$). The formula developed above has been proved in several ways, the one given below being quite commonly used.

Given the angles α and β, with their sum, $\alpha + \beta$, equal to angle XOP, the ordinate MP, and the other lines drawn as shown in Fig. 47.

$$\sin(\alpha + \beta) = \frac{MP}{OP} = \frac{MT + TP}{OP} = \frac{RS + TP}{OP}.$$

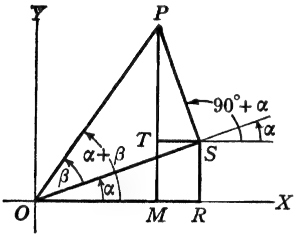


Fig. 47

But

$$RS = OS \sin\alpha,$$

$$OS = OP\cos\beta,$$

$$TP = SP \sin(90° + \alpha)$$

$$= SP\cos\alpha,$$

and

$$SP = OP \sin\beta.$$

Substituting in the expression for sin ($\alpha + \beta$) above;

$$\sin(\alpha + \beta) = \frac{OS}{OP}\sin\alpha + \frac{SP}{OP}\cos\beta$$

$$= \frac{OP\cos\beta\sin\alpha}{OP} + \frac{OP\cos\alpha\sin\beta}{OP},$$

$$\boldsymbol{\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta} \qquad [11]$$

While this formula has been proved only for $\alpha + \beta$ an acute angle, the method will apply when $\alpha + \beta$ is obtuse or larger.

One use of the formula for sin ($\alpha + \beta$) is shown below.

Example 1. If $\sin x = -\frac{5}{13}$, $180° < x < 270°$, and $\tan y = -\frac{4}{3}$, $90° < y < 180°$, find $\sin (x + y)$.

The angles are drawn and shown below. From these any required function of either angle may be found.

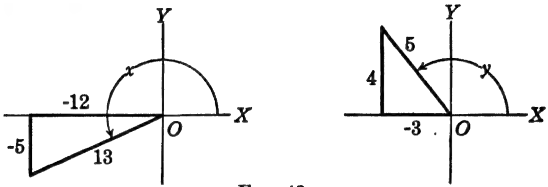


FIG. 48

$$\begin{aligned}\sin (x + y) &= \sin x \cos y + \cos x \sin y\\ &= \left(-\tfrac{5}{13}\right)\left(-\tfrac{3}{5}\right) + \left(-\tfrac{12}{13}\right)\left(\tfrac{4}{5}\right)\\ &= \tfrac{15}{65} - \tfrac{48}{65} = -\tfrac{33}{65}.\end{aligned}$$

The student should note that $\sin x + \sin y$ and $\sin (x + y)$ are different expressions. In the problem above,

$$\begin{aligned}\sin x + \sin y &= -\tfrac{5}{13} + \tfrac{4}{5} = \tfrac{27}{65};\\ \sin (x + y) &= -\tfrac{33}{65}.\end{aligned}$$

EXERCISES

1. Using the first method, prove the formula for $\sin (\alpha + \beta)$ where $270° < \alpha + \beta < 360°$.

2. Using the alternate method, prove the formula for $\sin (\alpha + \beta)$ where $90° < \alpha + \beta < 180°$.

Use the formula for $\sin (\alpha + \beta)$ *to determine the numerical value of the following:*

3. $\sin 90°$, using $90° = 45° + 45°$.

4. $\sin 90°$, using $90° = 60° + 30°$.

5. $\sin 75°$, using $75° = 45° + 30°$.

6. $\sin 105°$, using $105° = 60° + 45°$.

7. $\sin 165°$. **8.** $\sin 285°$. **9.** $\sin 345°$.

10. Given $\sin A = \frac{4}{5}$, $90° < A < 180°$, and $\cos B = \frac{5}{13}$, B an acute angle; find $\sin (A + B)$.

11. Given $\sin \alpha = -\frac{4}{5}$, $180° < \alpha < 270°$, $\cos \beta = -\frac{5}{13}$, $90° < \beta < 180°$; find $\sin (\alpha + \beta)$.

12. If $\alpha_3 = \text{arc} \tan \frac{3}{4}$ and $\beta_2 = \text{arc ctn} \left(-\frac{12}{5}\right)$, find $\sin (\alpha + \beta)$.

13. If $\alpha_2 = \text{arc} \tan \left(-\frac{8}{15}\right)$ and $\beta_3 = \text{arc} \cos (-0.8)$, find $\sin (\alpha + \beta)$.

14. Find which is the larger and by how much, $\sin (A + B)$ or $\sin A + \sin B$ if $A_3 = \tan^{-1} \frac{24}{7}$ and $B_3 = \text{arc} \tan \frac{4}{3}$.

Express as the sine of one angle:

15. $\sin A \cos 2A + \cos A \sin 2A$.

16. $\sin 2B \cos B + \cos 2B \sin B$.

17. Find one value of B if

$$\sin (C + D) \cos C + \cos (C + D) \sin C = \sin B.$$

18. If $\sin B = \frac{5}{13}$, $90° < B < 180°$, express $\frac{5}{13} \cos x - \frac{12}{13} \sin x$ as a function of $(x + B)$.

35. The cosine of the sum of two angles. From the relation, $\sin (90° + \theta) = \cos \theta$, and [11], a formula for $\cos (\alpha + \beta)$ may be derived.

$$\begin{aligned} \cos (\alpha + \beta) &= \sin [90° + (\alpha + \beta)] \\ &= \sin [(90° + \alpha) + \beta] \\ &= \sin (90° + \alpha) \cos \beta + \cos (90° + \alpha) \sin \beta. \end{aligned}$$

But

$$\sin (90° + \alpha) = \cos \alpha \quad \text{and} \quad \cos (90° + \alpha) = -\sin \alpha.$$

Hence

$$\mathbf{\cos (\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta.} \qquad [12]$$

Formula [12] may be stated in words as follows:

The cosine of the sum of two angles is equal to the product of their cosines minus the product of their sines.

As the proof depends only on two theorems which are true for all values of the angles involved, this theorem is true for all values of α and β.

36. The sine and cosine of the difference of two angles. Since $\sin (\alpha' + \beta') = \sin \alpha' \cos \beta' + \cos \alpha' \sin \beta'$ is true for all values of α' and β', let $\alpha' = \alpha$ and $\beta' = -\beta$. Then

$$\sin [\alpha + (-\beta)] = \sin \alpha \cos (-\beta) + \cos \alpha \sin (-\beta).$$

But

$$\cos(-\beta) = \cos\beta \quad \text{and} \quad \sin(-\beta) = -\sin\beta.$$

Hence

$$\boldsymbol{\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta.} \qquad [13]$$

Likewise from

$$\cos(\alpha' + \beta') = \cos\alpha'\cos\beta' - \sin\alpha'\sin\beta',$$
$$\cos(\alpha - \beta) = \cos\alpha\cos(-\beta) - \sin\alpha\sin(-\beta),$$

or

$$\boldsymbol{\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.} \qquad [14]$$

EXERCISES

1. Prove the formula for $\cos(\alpha + \beta)$ by using $\cos(\alpha + \beta) = \sin[90° - (\alpha + \beta)]$.

Given $\sec x = -\frac{5}{4}$, $90° < x < 180°$, $\tan y = \frac{3}{4}$, $180° < y < 270°$, find:

2. $\sin(x + y)$.

3. $\sin(x - y)$.

4. $\cos(x + y)$.

5. $\cos(x - y)$.

Given $A_3 = \sin^{-1}(-\frac{24}{25})$ and $B_2 = \cos^{-1}(-\frac{5}{13})$, find:

6. $\sin(A + B)$.

7. $\cos(A + B)$.

8. $\sin(A - B)$.

9. $\cos(A - B)$.

10. Using Fig. 47, develop the formula for $\cos(\alpha + \beta)$.

Find one value for x in each of the following:

11. $\sin 3A\cos A + \cos 3A\sin A = \sin x$.

12. $\sin 6C\sin 2C + \cos 6C\cos 2C = \cos 3x$.

13. $\sin 2A\cos 4A - \cos 2A\sin 4A = \sin x$.

14. $\cos 4B\cos 2B + \sin 2B\sin 4B = \cos 2x$.

15. Prove: $\dfrac{\sin(x + y) + \sin(x - y)}{\cos(x + y) + \cos(x - y)} = \tan x$.

16. Prove: $\dfrac{\sin(x - y) - \sin(x + y)}{\cos(x + y) - \cos(x - y)} = \operatorname{ctn} x$.

Without using tables find the value:

17. $\sin[\text{arc}\cos_1 \frac{1}{2} + \text{arc}\tan_2(-\frac{3}{4})]$.

18. $\cos[\text{arc}\sin_2(-\frac{1}{2}) + \text{arc}\tan_3 \frac{3}{5}]$.

19. $\sin\,[\text{arc}\sec_4 \frac{5}{4} - \text{arc}\sin_2 \frac{12}{13}]$.

20. $\cos\,[\text{arc}\cos_4 \frac{1}{3} - \text{arc}\cos_1 \frac{1}{4}]$.

21. Express $\sin 3x$ in terms of $\sin x$.

HINT. Take $\sin 3x = \sin(2x + x)$, and in the next step use $2x = x + x$.

22. Prove $\cos 3x = 4\cos^3 x - 3\cos x$.

37. The tangent of the sum and of the difference of two angles. From the formulas for $\sin(\alpha + \beta)$ and $\cos(\alpha + \beta)$ a formula for $\tan(\alpha + \beta)$ may be derived:

$$\tan(\alpha + \beta) = \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)}$$

$$= \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}.$$

Dividing each term of the numerator and denominator by $\cos\alpha\cos\beta$ and simplifying,

$$\tan(\alpha + \beta) = \frac{\dfrac{\sin\alpha\cancel{\cos\beta}}{\cos\alpha\cancel{\cos\beta}} + \dfrac{\cancel{\cos\alpha}\sin\beta}{\cancel{\cos\alpha}\cos\beta}}{\dfrac{\cancel{\cos\alpha}\cancel{\cos\beta}}{\cancel{\cos\alpha}\cancel{\cos\beta}} - \dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}},$$

$$\boldsymbol{\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}}. \qquad [15]$$

Stated in words [15] becomes:

The tangent of the sum of two angles is equal to the sum of their tangents divided by the quantity, one minus the product of their tangents.

Since the equation $\tan(\alpha' + \beta') = \dfrac{\tan\alpha' + \tan\beta'}{1 - \tan\alpha'\tan\beta'}$, is true for all values of α' and β', let $\alpha' = \alpha$ and $\beta' = -\beta'$. Then

$$\tan(\alpha - \beta) = \frac{\tan\alpha + \tan(-\beta)}{1 - \tan\alpha\tan(-\beta)},$$

or since $\tan(-\beta) = -\tan\beta$,

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}. \qquad [16]$$

Formulas [15] and [16] may be used to find the algebraic sum of two angles expressed by inverse functions.

Example 1. Without using tables to find the angles, find the value of $\tan^{-1}\frac{1}{3} + \tan^{-1}(-1)$.

Let $A = \tan^{-1}\frac{1}{3}$ and $B = \tan^{-1}(-1)$, and let $A + B = C$. Then C is required and as the given angles are defined by inverse functions, the sum may be expected to appear as an inverse function. Any function of $A + B$ equals the same function of C. As the formula for $\tan(A + B)$ involves only the given functions of A and B, it will be better to take the tangent of both sides of the equation $A + B = C$.

$$\tan C = \tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$= \frac{\frac{1}{3} - 1}{1 + \frac{1}{3}} = -\tfrac{1}{2}.$$

Hence $$C = \tan^{-1}(-\tfrac{1}{2})$$

and $$\tan^{-1}\tfrac{1}{3} + \tan^{-1}(-1) = \tan^{-1}(-\tfrac{1}{2}).$$

EXERCISES

1. Prove the formula for $\tan(\alpha - \beta)$ from $\tan(\alpha - \beta) = \dfrac{\sin(\alpha - \beta)}{\cos(\alpha - \beta)}$.

2. Derive a formula for $\operatorname{ctn}(\alpha + \beta)$ in terms of $\operatorname{ctn}\alpha$ and $\operatorname{ctn}\beta$.

Given $\sin\alpha = \frac{3}{5}$, $90° < \alpha < 180°$ and $\cos\beta = -\frac{8}{17}$, $180° < \beta < 270°$, evaluate:

3. $\tan(\alpha + \beta)$. **4.** $\tan(\alpha - \beta)$. **5.** $\tan(\beta - \alpha)$.

6. By what must $\tan x + \tan y$ be divided to give $\tan(x + y)$?

7. Determine whether $\tan(A + B)$ and $\tan A + \tan B$ are equal if (a) $A = 30°$, $B = 30°$; (b) $A = 135°$, $B = 45°$; (c) $A = 150°$, $B = 240°$; (d) $A = 180°$, $B = 0°$.

Given $\tan A = \frac{2}{3}$, $\text{ctn}\, B = \frac{1}{3}$, evaluate:

8. $\tan (A + B)$. **9.** $\tan (B - A)$. **10.** $\text{ctn}\, (A - B)$.

11. If $A = \tan^{-1} \frac{1}{2}$, $B = \cos^{-1} \frac{4}{5}$, find the value of $\tan (A + B)$.

12. If $x = \text{ctn}^{-1} \frac{1}{2}$, $y = \sin^{-1} (-\frac{5}{13})$, find the value of $\tan (x - y)$.

13. Find $\tan [\tan^{-1} \frac{2}{3} - \tan^{-1} (-\frac{1}{2})]$.

14. Find $\tan [\tan^{-1} \frac{1}{2} + \text{ctn}^{-1} \frac{4}{3}]$.

Without tables find the algebraic sum of the following, expressing the result as an inverse tangent:

15. $\text{arc} \tan_1 2 - \text{arc} \tan_2 (-\frac{1}{2})$.

16. $\text{arc}\, \text{ctn}_3 \frac{4}{3} + \text{arc} \cos_3 (-\frac{5}{13})$.

17. $\text{arc} \tan_2 (-\frac{1}{3}) + \text{arc} \sin_2 \frac{4}{5}$.

18. $\text{arc} \sin_1 \frac{1}{2} - \text{arc} \cos_2 \left(\frac{-\sqrt{3}}{2}\right)$.

Prove the following identities:

19. $\tan (45° + \theta) = (1 + \tan \theta) \div (1 - \tan \theta)$.

20. $\dfrac{\tan (x + y) - \tan x}{1 + \tan x \tan (x + y} = \tan y$.

21. $\dfrac{\sin (x - y)}{\cos (x + y)} = \dfrac{\tan x - \tan y}{1 - \tan x \tan y}$.

22. $\dfrac{\tan (45° - x)}{\tan (45° + x)} = \dfrac{\sec^2 x - 2 \tan x}{\sec^2 x + 2 \tan x}$.

38. Functions of double angles. In the formulas for $\sin (\alpha + \beta)$, $\cos (\alpha + \beta)$, and $\tan (\alpha + \beta)$, let $\beta = \alpha$. Then

$$\sin (\alpha + \alpha) = \sin \alpha \cos \alpha + \cos \alpha \sin \alpha,$$

$$\mathbf{\sin 2\, \alpha = 2 \sin \alpha \cos \alpha;} \qquad [17]$$

$$\cos (\alpha + \alpha) = \cos \alpha \cos \alpha - \sin \alpha \sin \alpha,$$

$$\mathbf{\cos 2\, \alpha = \cos^2 \alpha - \sin^2 \alpha.} \qquad [18a]$$

$$\mathbf{\cos 2\, \alpha = 1 - \sin^2 \alpha - \sin^2 \alpha = 1 - 2 \sin^2 \alpha.} \qquad [18b]$$

$$\mathbf{\cos 2\, \alpha = \cos^2 \alpha - (1 - \cos^2 \alpha) = 2 \cos^2 \alpha - 1.} \qquad [18c]$$

$$\tan (\alpha + \alpha) = \frac{\tan \alpha + \tan \alpha}{1 - \tan \alpha \tan \alpha}$$

$$\mathbf{\tan 2\, \alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha}.} \qquad [19]$$

These formulas may also be stated in words:

The sine of twice an angle is equal to twice the product of the sine of the angle by its cosine.

The cosine of twice an angle is equal to the square of the cosine of the angle minus the square of its sine.

The tangent of twice an angle is equal to twice its tangent divided by the quantity, one minus the square of the tangent.

In these formulas the angles on the right-hand side of the equation are half those on the left. Hence the equations may also be written:

$$\sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}, \quad \text{and} \quad \sin 3\theta = 2\sin\frac{3\theta}{2}\cos\frac{3\theta}{2};$$

$$\cos\alpha = \cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}, \quad \text{and} \quad \cos\frac{3\alpha}{2} = \cos^2\frac{3\alpha}{4} - \sin^2\frac{3\alpha}{4};$$

$$\tan\alpha = \frac{2\tan\frac{\alpha}{2}}{1-\tan^2\frac{\alpha}{2}}, \quad \text{and} \quad \tan\frac{2\beta}{3} = \frac{2\tan\frac{\beta}{3}}{1-\tan^2\frac{\beta}{3}}.$$

Students should practice writing these formulas for other values of the angle.

39. Functions of half-angles. In the identities,

$$\sin^2\theta + \cos^2\theta = 1 \quad \text{and} \quad \cos 2\theta = \cos^2\theta - \sin^2\theta,$$

let $\theta = \frac{\alpha}{2}$. Then

$$\sin^2\frac{\alpha}{2} + \cos^2\frac{\alpha}{2} = 1 \tag{1}$$

and

$$\cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2} = \cos\alpha. \tag{2}$$

By subtraction of (2) from (1),

$$2\sin^2\frac{\alpha}{2} = 1 - \cos\alpha$$

or

$$\sin^2\frac{\alpha}{2} = \frac{1 - \cos\alpha}{2}.$$

$$\therefore \sin\frac{\alpha}{2} = +\sqrt{\frac{1 - \cos\alpha}{2}} \text{ or } \sin\frac{\alpha}{2} = -\sqrt{\frac{1 + \cos\alpha}{2}}. \qquad [20]$$

By addition of (1) and (2) above,

$$2\cos^2\frac{\alpha}{2} = 1 + \cos\alpha$$

or

$$\cos^2\frac{\alpha}{2} = \frac{1 + \cos\alpha}{2}.$$

$$\therefore \cos\frac{\alpha}{2} = +\sqrt{\frac{1 + \cos\alpha}{2}} \quad\text{or}\quad \cos\frac{\alpha}{2} = -\sqrt{\frac{1 + \cos\alpha}{2}}. \qquad [21]$$

Dividing [20] by [21]

$$\tan\frac{\alpha}{2} = +\sqrt{\frac{1 - \cos\alpha}{1 + \cos\alpha}} \text{ or } \tan\frac{\alpha}{2} = -\sqrt{\frac{1 - \cos\alpha}{1 + \cos\alpha}}. \qquad [22a]$$

From [22a] two simple formulas for $\tan\frac{\alpha}{2}$ may be derived:

$$\tan\frac{\alpha}{2} = \sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} = \frac{\sqrt{1-\cos\alpha}}{\sqrt{1+\cos\alpha}} \cdot \frac{\sqrt{1-\cos\alpha}}{\sqrt{1-\cos\alpha}} = \frac{1-\cos\alpha}{\sqrt{1-\cos^2\alpha}}.$$

$$\tan\frac{\alpha}{2} = \frac{1 - \cos\alpha}{\sin\alpha}. \qquad [22b]$$

$$\tan\frac{\alpha}{2} = \sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} = \frac{\sqrt{1-\cos\alpha}}{\sqrt{1+\cos\alpha}} \cdot \frac{\sqrt{1+\cos\alpha}}{\sqrt{1+\cos\alpha}} = \frac{\sqrt{1-\cos^2\alpha}}{1+\cos\alpha}.$$

$$\tan\frac{\alpha}{2} = \frac{\sin\alpha}{1 + \cos\alpha}. \qquad [22c]$$

It can be shown that the sign on the right-hand side of [22b] and [22c] is always positive regardless of whether the positive or negative radical is used in [22a].

If α is a given angle there is only one value for $\sin\frac{\alpha}{2}$, hence formula [20] is not written with a $\pm$ sign as that sign implies that either of two values satisfy an equation. This also applies to formulas [21] and [22a]. In each problem involving these formulas it is necessary to determine whether to use the equation involving the plus sign or that involving the minus sign. A method for determining the sign is shown in Example 1 below.

Example 1. Given $\sec C = \frac{5}{4}$, $270° < C < 360°$, find $\sin\frac{C}{2}$, $\cos\frac{C}{2}$, and $\tan\frac{C}{2}$

Formulas [20], [21], and [22c] apply here and it is necessary to determine the sign in using [20] and [21].

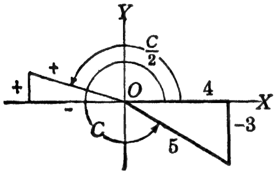


FIG. 49

Since

$$270° < C < 360°,$$

$$135° < \frac{C}{2} < 180°.$$

As $\frac{C}{2}$ lies in the second quadrant, $\sin\frac{C}{2}$ is positive, $\cos\frac{C}{2}$ is negative, but the sign of $\tan\frac{C}{2}$ does not need to be determined independent of formula [22c].

For this particular problem:

By [20] $\quad \sin\frac{C}{2} = +\sqrt{\frac{1-\cos C}{2}} = \sqrt{\frac{1-\frac{4}{5}}{2}} = \sqrt{\frac{1}{10}} = \frac{1}{10}\sqrt{10}.$

By [21] $\quad \cos\frac{C}{2} = -\sqrt{\frac{1+\cos C}{2}} = -\sqrt{\frac{1+\frac{4}{5}}{2}} = -\sqrt{\frac{9}{10}} = -\frac{3}{10}\sqrt{10}.$

By [22c] $\quad \tan\frac{C}{2} = \frac{\sin C}{1+\cos C} = \frac{-\frac{3}{5}}{1+\frac{4}{5}} = \frac{-\frac{3}{5}}{\frac{9}{5}} = -\frac{1}{3}.$

Either [22a] or [22b] will give the same result for $\tan\frac{C}{2}$.

Example 2. Find the value of $\cos\left[\frac{1}{2}\text{ arc }\cos_2(-\frac{1}{3})+\frac{\pi}{2}\right]\cdot$

Let $A = \text{arc }\cos_2(-\frac{1}{3})$. The problem then is one of evaluating $\cos\left(\frac{A}{2}+\frac{\pi}{2}\right)$. But $\cos\left(\frac{\pi}{2}+\theta\right) = -\sin\theta$, hence

$$\cos\left(\frac{A}{2}+\frac{\pi}{2}\right) = -\sin\frac{A}{2}\cdot$$

Now $90° < A < 180°$, $45° < \frac{A}{2} < 90°$, and $\sin\frac{A}{2} = +\sqrt{\frac{1-\cos A}{2}}\cdot$ Hence

$$-\sin\frac{A}{2} = -\sqrt{\frac{1-(-\frac{1}{3})}{2}} = -\sqrt{\frac{2}{3}} = -\frac{1}{3}\sqrt{6}.$$

$\cos\left(\frac{A}{2}+\frac{\pi}{2}\right)$ could also have been expanded by using the formula for $\cos(\alpha+\beta)$.

EXERCISES

Given $\sin\theta = \frac{24}{25}$, $90° < \theta < 180°$, find the value:

1. $\sin 2\theta$. **3.** $\tan 2\theta$. **5.** $\cos\frac{\theta}{2}\cdot$

2. $\sin\frac{\theta}{2}\cdot$ **4.** $\tan\frac{\theta}{2}\cdot$ **6.** $\cos 2\theta$.

Given $\theta = \tan^{-1}\frac{15}{8}$, $180° < \theta < 360°$, find the value:

7. $\sin 2\theta$. **9.** $\tan 2\theta$. **11.** $\sin\frac{\theta}{2}\cdot$

8. $\cos\frac{\theta}{2}\cdot$ **10.** $\tan\frac{\theta}{2}\cdot$ **12.** $\cos 2\theta$.

Given $\cos\theta = -\frac{4}{5}$, $180° < \theta < 270°$, find the value:

13. $\sin\left(\frac{\pi}{2}+2\theta\right)$. **14.** $\cos\left(\pi+\frac{\theta}{2}\right)$. **15.** $\tan\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\cdot$

Find a value of x in each of the following:

16. $\sin 2A = \sqrt{\frac{1-\cos x}{2}}\cdot$ **18.** $2\sin 2x\cos 2x = \sin 3A$.

17. $\cos\frac{A}{2} = \sqrt{\frac{1+\cos x}{2}}$. **19.** $\tan A = \frac{\sin x}{1+\cos x}\cdot$

20. $2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} = \sin \frac{x}{2}$. **21.** $\cos^2 (4A) - \sin^2 (4A) = \cos x$.

Evaluate:

22. $\sin \left[2 \text{ arc } \sin_4 \frac{5}{13} + \frac{\pi}{2} \right]$.

23. $\cos \left[2 \text{ arc } \tan_4 (-0.75) + \frac{\pi}{2} \right]$.

24. $\sin [\frac{1}{2} \text{ arc } \cos_3 (-\frac{3}{5}) + \pi]$.

25. $\cos [\pi - \frac{1}{2} \text{ arc } \cos_3 (-\frac{4}{5})]$.

Use the formulas of Art. 39 *to find the sine, cosine, and tangent of the following angles:*

26. $15°$. **27.** $22\frac{1}{2}°$. **28.** $75°$. **29.** $105°$.

If $\sin 4x = a$ and $\cos 4x = b$, find the value of:

30. $\sin 8x$. **32.** $\cos 8x$. **34.** $\tan 8x$.
31. $\sin 2x$. **33.** $\cos 2x$. **35.** $\tan 2x$.

Prove the following identities:

36. $\sin^4 x - \cos^4 x = -\cos 2x$.

37. $\tan (45° + x) = \dfrac{1 + \sin 2x}{\cos 2x}$.

38. $\dfrac{\cos 2x}{1 + \sin 2x} = \tan (45° - x)$.

39. $\sin^2 \dfrac{x}{2} = \dfrac{\sec x - 1}{2 \sec x}$.

40. $\sin 4x + \cos 4x = \cos^2 2x + 2 \sin 2x \cos 2x - \sin^2 2x$.

41. $\dfrac{\sin x + \cos x}{\cos x - \sin x} - \dfrac{\cos x - \sin x}{\cos x + \sin x} = 2 \tan 2x$.

42. $\sin^2 \theta \cos^2 \theta = \dfrac{1 - \cos 4\theta}{8}$.

43. $\dfrac{\tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} = \frac{1}{2} \sin x$.

44. $\tan \dfrac{\theta}{2} = \csc \theta - \text{ctn } \theta$.

45. $\text{ctn } \dfrac{\theta}{2} = \csc \theta + \text{ctn } \theta$.

40. The algebraic sum of sines and cosines. It may be desired to express the algebraic sum of two sines or of two cosines as a product, or inversely, the product of sines and cosines as a sum. From the equations

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta,$$

and

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta,$$

addition gives

$$\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2\sin\alpha\cos\beta.$$

Letting

$$\alpha + \beta = P \quad \text{and} \quad \alpha - \beta = Q,$$

addition and subtraction gives

$$2\alpha = P + Q \quad \text{or} \quad \alpha = \tfrac{1}{2}(P + Q)$$

and

$$2\beta = P - Q \quad \text{or} \quad \beta = \tfrac{1}{2}(P - Q).$$

Hence

$$\mathbf{\sin P + \sin Q = 2\sin\tfrac{1}{2}(P + Q)\cos\tfrac{1}{2}(P - Q).} \qquad [23]$$

By subtracting the expression for $\sin(\alpha - \beta)$ from that for $\sin(\alpha + \beta)$ and going through a process similar to that above,

$$\mathbf{\sin P - \sin Q = 2\cos\tfrac{1}{2}(P + Q)\cos\tfrac{1}{2}(P - Q).} \qquad [24]$$

Likewise by using

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

and

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta,$$

there can be derived

$$\cos(\alpha + \beta) + \cos(\alpha - \beta) = 2\cos\alpha\cos\beta,$$
$$\cos(\alpha + \beta) - \cos(\alpha - \beta) = -2\sin\alpha\sin\beta.$$

Therefore

$$\cos P + \cos Q = 2\cos\tfrac{1}{2}(P+Q)\cos\tfrac{1}{2}(P-Q). \qquad [25]$$

$$\cos P - \cos Q = -2\sin\tfrac{1}{2}(P+Q)\sin\tfrac{1}{2}(P-Q). \qquad [26]$$

The use of these formulas to transform sums to products and products to sums is shown in the examples following.

Example 1. Express $\sin 4x - \sin 2x$ as the product of trigonometric functions.

Using [24] and taking $P = 4x$, $Q = 2x$.

$$\begin{aligned}\sin 4x - \sin 2x &= 2\cos\tfrac{1}{2}(4x+2x)\sin\tfrac{1}{2}(4x-2x)\\ &= 2\cos 3x\sin x.\end{aligned}$$

Example 2. Express $\sin 4x \sin 2x$ as the algebraic sum of sines or cosines.

From [26], $\sin\frac{1}{2}(P+Q)\sin\frac{1}{2}(P-Q) = -\frac{1}{2}(\cos P - \cos Q)$.

Let

$$\tfrac{1}{2}(P+Q) = 4x \quad\text{and}\quad \tfrac{1}{2}(P-Q) = 2x.$$

By addition of the equations, $P = 6x$, and by subtraction, $Q = 2x$.

$$\therefore\ \sin 4x\sin 2x = -\tfrac{1}{2}(\cos 6x - \cos 2x).$$

Example 3. Simplify: $\dfrac{\sin 5x - \sin 3x}{\cos 4x}$.

$$\begin{aligned}\frac{\sin 5x - \sin 3x}{\cos 4x} &= \frac{2\cos\frac{1}{2}(5x+3x)\sin\frac{1}{2}(5x-3x)}{\cos 4x}\\ &= \frac{2\cos 4x\sin x}{\cos 4x}\\ &= 2\sin x.\end{aligned}$$

EXERCISES

1. Write out the complete proof for [25] and [26].

Express as the product of trigonometric functions:

2. $\sin 4\theta + \sin 2\theta$.

3. $\sin 4\theta - \sin 2\theta$.

4. $\cos 4\theta - \cos 2\theta$.

5. $\cos 2\theta + \cos 4\theta$.

6. $\cos 3\theta + \cos 5\theta$.

7. $\sin 3\theta - \sin 2\theta$.

8. $\sin 3\theta + \sin 2\theta$.

9. $\sin 2\theta - \sin 3\theta$.

10. $\cos 3\theta - \cos 5\theta$.

11. $\cos 6\theta + \cos 2\theta$.

12. $\sin 2\theta + \sin 4\theta$.

13. $\sin 2\theta - \sin 4\theta$.

14. $\sin\theta + \sin 5\theta$.

15. $\cos 8\theta - \cos 3\theta$.

16. $\cos 3\theta - \cos 7\theta$.

Prove the following:

17. $\sin 260° + \sin 80° = 0.$

18. $\sin 316° - \sin 136° = 2 \cos 226°.$

19. $\cos 108° - \cos 288° = 2 \sin 198°.$

20. $\cos 252° + \cos 108° = -2 \cos 72°.$

Evaluate:

21. $\sin 75° - \sin 15°.$

22. $\sin 195° + \sin 75°.$

23. $\cos 75° + \cos 15°.$

24. $\cos 195° + \sin 165°.$

Express as the algebraic sum of sines or cosines:

25. $\sin 4x \cos 2x.$

26. $\cos \frac{x}{2} \sin \frac{3x}{2}.$

27. $\cos 8\theta \sin 2\theta.$

28. $\cos \frac{5y}{2} \cos \frac{y}{2}.$

29. $\sin \frac{B}{2} \sin \frac{3B}{2}.$

30. $\cos \frac{x}{2} \sin \frac{5x}{2}.$

31. $\cos 6x \sin 2x.$

32. $\sin A \sin \frac{A}{2}.$

33. $\cos \frac{3x}{2} \cos \frac{5x}{2}.$

34. $\sin x \sin 3x.$

35. $\cos \frac{5x}{2} \sin \frac{3x}{2}.$

36. $\sin 4\theta \sin 2\theta.$

37. $\cos 2B \cos 4B.$

38. $\cos \theta \cos 5\theta.$

39. $\sin 2\alpha \cos 3\alpha.$

Without using tables, find the value of the following:

40. $\sin 15° \cos 75°.$

41. $\cos 15° \cos 75°.$

42. $\cos 15° \sin 75°.$

43. $\sin 15° \sin 165°.$

44. $\sin 165° \cos 105°.$

45. $\cos 165° \sin 105°.$

Prove the following identities:

46. $\dfrac{\cos 5\theta - \cos 3\theta}{\sin 3\theta + \sin 5\theta} = -\tan \theta.$

47. $\dfrac{\cos 2\theta - \cos 4\theta}{\sin 2\theta + \sin 4\theta} = \tan \theta.$

48. $\dfrac{\sin 6\theta - \sin 2\theta}{\cos 6\theta - \cos 2\theta} = -\operatorname{ctn} 4\theta.$

49. $\dfrac{\sin 2\theta - \sin 3\theta}{\cos 3\theta - \cos 2\theta} = \operatorname{ctn} \dfrac{5\theta}{2}.$

50. $\dfrac{\cos (x + 3y) + \cos (x - 3y)}{\cos x} = 2 \cos 3y.$

51. $\dfrac{\sin (2x + y) - \sin (2x - y)}{\sin 4x} = \dfrac{\sin y}{\sin 2x}.$

52. $\dfrac{\sin 4x - 2 \sin 3x + \sin 2x}{\sin 3x} = -4 \sin^2 \dfrac{x}{2}.$

53. $\dfrac{\cos 6x + 2\sin 4x - \cos 2x}{2\sin 4x} = 1 - \sin 2x.$

54. $\dfrac{\sin 5x - 2\sin 3x + \sin x}{\cos 5x - \cos x} = \tan x.$

41. Trigonometric equations. Trigonometric equations have been defined in Art. 31, and some simple ones solved. The formulas of this chapter permit more extended transformations of given equations and allow more methods of solution. The methods of solving algebraic equations are applicable as these equations are algebraic in form. Some solutions are shown by the examples following.

Example 1. Find the values of x from 0° to 360° inclusive that satisfy $2\sin^2 2x + \cos 2x = 0$.

Transforming to one function of the unknown angle,

$$2 - 2\cos^2 2x + \cos 2x = 0,$$
$$2\cos^2 2x - \cos 2x - 2 = 0.$$

Solving by the quadratic formula,

$$\cos 2x = \frac{1 \pm \sqrt{17}}{4} = 1.28078 \text{ or } -0.78078.$$

Hence the given equation is satisfied when (a) $\cos 2x = 1.28078$; (b) $\cos 2x = -0.78078$.

In case (a): There is no value of x that satisfies this equation.

In case (b): arc cos (0.78078) = 38° 40′.1, from tables. Hence the given equation is satisfied when

$$2x = 141°\,19'.9 \quad \text{or} \quad 218°\,40'.1,$$

or any angle coterminal with them. These values of $2x$ are given by

$$2x = 141°\,19'.9 + n \cdot 360° \text{ and } 2x = 218°\,40'.1 + n \cdot 360°,$$

where n is any integer, positive or negative.

$$\therefore x = 70°\,40'.0 + n \cdot 180° \quad \text{or} \quad 109°\,20'.1 + n \cdot 180°.$$

The problem then becomes one of assigning to n such values as will give values of x between 0° and 360°.

For $n = 0$, $x = 70°\,40'.0$, $109°\,20'.1$;
$n = 1$, $x = 250°\,40'.0$, $289°\,20'.1$.

By trial it will be found that no other value of n will give values of x between 0° and 360°.

The essential details of the solution, after the values of $\cos 2x$ have been found, are exhibited in the following table.

$\cos 2x = 1.28078$, or	$\cos 2x = -0.78078$.
There is no value of x that satisfies this equation.	$\text{arc cos }(0.78078) = 38° 40'.1.$ $\text{arc cos }(-0.78078) = 180° \pm 38° 40'.1.$ $\therefore 2x = 141° 19'.9 + n \cdot 360°,$ or $2x = 218° 40'.1 + n \cdot 360°.$ $x = 70° 40'.0 + n \cdot 180°$, $x = 109° 20'.1 + n \cdot 180°.$

n	0	1
x	70° 40'.0	250° 40'.0
x	109° 20'.1	289° 20'.1

Example 2. Solve $6 \sin \theta - 3 \cos \theta = 2$ for all positive angles less than 360°.

For $\cos \theta$ substitute $\sqrt{1 - \sin^2 \theta}$, giving

$$6 \sin \theta - 3 \sqrt{1 - \sin^2 \theta} = 2,$$
$$6 \sin \theta - 2 = 3 \sqrt{1 - \sin^2 \theta},$$
$$36 \sin^2 \theta - 24 \sin \theta + 4 = 9 - 9 \sin^2 \theta,$$
$$45 \sin^2 \theta - 24 \sin \theta - 5 = 0.$$
$$\sin \theta = \frac{24 \pm \sqrt{1476}}{90} = 0.69354 \text{ or } -0.16021.$$

For these values of $\sin \theta$,

$$\theta = 43° 54'.7,\ 136° 5'.3,\ 189° 13'.7,\ 350° 46'.8.$$

These angles must be tested by substitution in the original equation as both sides of the equation were squared in the solution. By trial it will be found that the equation is satisfied only by

$$\theta = 43° 54'.7 \text{ and } 189° 13'.2.$$

Another solution of this same equation is shown in the next Article where several special types of trigonometric equations are discussed.

42. Special types of trigonometric equations. Three common types of trigonometric equations and the devices used in their solution are shown below. The types are first mentioned, the solution of one or more examples of each type is shown and a discussion given.

Type 1. $a \sin x + b \cos x = c, \quad c \leqq \sqrt{a^2 + b^2}.$

Example 1. Solve $6 \sin \theta - 3 \cos \theta = 2$ for all positive angles less than 360°.

Comparing with the type equation,

$$a = 6,\ b = -3,\ c = 2, \text{ and } \sqrt{a^2 + b^2} = 3\sqrt{5}.$$

Dividing both members of the equation by $3\sqrt{5}$ gives

$$\frac{6}{3\sqrt{5}} \sin \theta - \frac{3}{3\sqrt{5}} \cos \theta = \frac{2}{3\sqrt{5}}.$$

Let $\frac{6}{3\sqrt{5}} = \sin \beta$, and β be acute, then $\frac{3}{3\sqrt{5}} = \cos \beta$, as shown in Fig. 50c. By substituting in the equation above,

$$\sin \beta \sin \theta - \cos \beta \cos \theta = \frac{2}{3\sqrt{5}}.$$

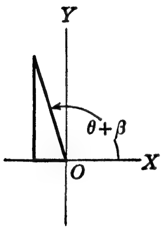


FIG. 50a

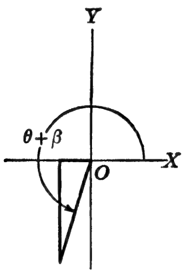


FIG. 50b

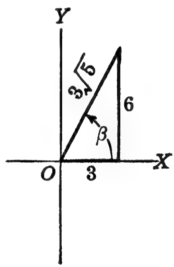


FIG. 50c

The left-hand side of this equation is in the form of the cosine of the difference of two angles. Rewriting the equation and, for future

convenience, placing the unknown angle θ first,

$$\cos\theta\cos\beta - \sin\theta\sin\beta = -\frac{2}{3\sqrt{5}} = -\frac{2\sqrt{5}}{15},$$

or

$$\cos(\theta+\beta) = -\frac{2\sqrt{5}}{15} = -0.29814.$$

$$\cos_1^{-1}(0.29814) = 72°\ 39'.3,$$

$$\beta = \tan_1^{-1}(2) = 63°\ 26'.1.$$

$\theta+\beta$	107° 20′ 7	252° 39′.3
β	63° 26′.1	63° 26′.1
θ	43° 54′.6	189° 13′.2

The general method of solving problems of this type will next be discussed. The first step is to divide both members of the equation by $\sqrt{a^2+b^2}$, then

$$\frac{a}{\sqrt{a^2+b^2}}\sin x + \frac{b}{\sqrt{a^2+b^2}}\cos x = \frac{c}{\sqrt{a^2+b^2}}.$$

Let $\dfrac{a}{\sqrt{a^2+b^2}} = \sin\beta$, and β be an acute angle, then

$$\cos\beta = \sqrt{1-\sin^2\beta} = \sqrt{1-\frac{a^2}{a^2+b^2}} = \sqrt{\frac{b^2}{a^2+b^2}} = \frac{b}{\sqrt{a^2+b^2}}.$$

The equation may then be written

$$\sin\beta\sin x + \cos\beta\cos x = \frac{c}{\sqrt{a^2+b^2}}$$

or

$$\cos(x-\beta) = \frac{c}{\sqrt{a^2+b^2}}.$$

To have a solution it is necessary that $c \leqq \sqrt{a^2+b^2}$. The numerical values of a, b, and c, with tables, give the values for $(x-\beta)$, and β can be found from the equation where it was defined. Hence x can be found.

Equations of this type can also be solved by taking $\frac{a}{\sqrt{a^2+b^2}} = \cos\beta$, and it is not necessary to impose the restriction that β be an acute angle. The equation above has also been solved in the preceding article. It may be interesting to compare the answers. The difference between them illustrates the approximate nature of the results of logarithmic calculation.

Type 2. $\sin ax = \cos bx$, or $\tan ax = \text{ctn}\, bx$.

The method of solving equations of this type will be shown by illustrative problems.

Example 2. Solve $\tan 3\theta = \text{ctn}\, 2\theta$ for all values of θ from 0° to 360° inclusive.

As $\text{ctn}\, 2\theta$ is identically equal to $\tan(90° - 2\theta)$, the equation can also be written

$$\tan 3\theta = \tan(90° - 2\theta).$$

Any angle coterminal with $90° - 2\theta$ has the same tangent as $90° - 2\theta$. Likewise $180° + (90° - 2\theta)$, or any angle coterminal with it, has the same tangent as $90° - 2\theta$. As no other angles have this same tangent, there can be formed equations in θ which are satisfied by the same values of θ, and these only, as the given equation. The given equation is satisfied when either

$$3\theta = 90° - 2\theta + n \cdot 360°, \quad \text{or} \quad 3\theta = 180° + 90° - 2\theta + n \cdot 360°.$$

$$5\theta = 90° + n \cdot 360°,$$
$$\theta = 18° + n \cdot 72°.$$

n	0	1	2	3	4
θ	18°	90°	162°	234°	306°

$$5\theta = 270° + n \cdot 360°,$$
$$\theta = 54° + n \cdot 72°.$$

n	0	1	2	3	4
θ	54°	126°	198°	270°	342°

By trial it will be found that no other value of n will give values of x between 0° and 360°.

Example 3. Solve $\sin 2\theta = \cos 3\theta$ for all values of θ from $0°$ to $360°$ inclusive.

First reduce the equation to one involving only one function by replacing $\cos 3\theta$ by its identically equal expression, $\sin(90° - 3\theta)$. This gives

$$\sin 2\theta = \sin(90° - 3\theta).$$

To form an equivalent equation in θ, write 2θ equal to $90° - 3\theta$ and to all other angles whose sine is equal to the sine of $90° - 3\theta$. All such angles are given by $90° - 3\theta + n \cdot 360°$ and $180° - (90° - 3\theta) + n \cdot 360°$, where n is any integer, positive or negative. Then the given equation is satisfied when

$$2\theta = 90 - 3\theta + n \cdot 360°, \qquad \text{or} \quad 2\theta = 180° - (90° - 3\theta) + n \cdot 360°.$$

$$5\theta = 90° + n \cdot 360°,$$
$$\theta = 18° + n \cdot 72°.$$

n	0	1	2	3	4
θ	18°	90°	162°	234°	306°

$$-\theta = 90° + n \cdot 360°,$$
$$\theta = -90° - n \cdot 360°.$$

n	0	-1	-2
θ	$-\cancel{90}°$	270°	$\cancel{630}°$

Replacements other than those used above could also have been made. Thus Example 2 could also have been solved by replacing $\tan 3\theta$ by $\operatorname{ctn}(90° - 3\theta)$, and Example 3 by either of the following; $\cos 3\theta$ by $\sin(90° + 3\theta)$, or $\sin 2\theta$ by $\cos(90° - 2\theta)$.

Type 3.

$$\sin ax + \sin bx + \sin cx = 0,$$
$$\cos ax + \cos bx + \cos cx = 0,$$
$$\sin ax + \sin bx + \cos cx = 0,$$
$$\cos ax + \cos bx + \sin cx = 0.$$

Equations of these types should suggest formulas [23] to [26], by which they may often be reduced to factorable forms.

Example 3. Solve $\sin\theta + \sin 3\theta + \sin 5\theta = 0$ for all values of θ from $0°$ to $360°$ inclusive.

Applying [23] to the first and third terms of the left-hand member,

$$2 \sin 3\theta \cos 2\theta + \sin 3\theta = 0,$$
$$\sin 3\theta\,(2 \cos 2\theta + 1) = 0.$$

The given equation is satisfied when

$\sin 3\theta = 0,$ or $2 \cos 2\theta + 1 = 0.$

$$\sin 3\theta = 0$$
$$3\theta = 0° + n \cdot 360°$$
or $$3\theta = 180° + n \cdot 360°.$$
$$\therefore\ \theta = n \cdot 120°$$
or $$\theta = 60° + n \cdot 120°.$$

When $\theta = n \cdot 120°$

n	0	1	2	3
θ	0°	120°	240°	360°

When $\theta = 60° + n \cdot 120°$

n	0	1	2
θ	60°	180°	300°

$$\cos \theta = -\tfrac{1}{2}$$
$$2\theta = 120° + n \cdot 360°$$
or $$2\theta = 240° + n \cdot 360°.$$
$$\therefore\ \theta = 60° + n \cdot 180°$$
or $$\theta = 120° + n \cdot 180°.$$

When $\theta = 60° + n \cdot 180°$

n	0	1
θ	60°	240°

When $\theta = 120° + n \cdot 180°$

n	0	1
θ	120°	300°

$\therefore$ The given equation is satisfied when $\theta = 0°, 60°, 120°, 180°, 240°, 300°, 360°$.

43. Simultaneous trigonometric equations. Simultaneous equations involving trigonometric functions can often be solved. One such system is solved below.

Example 1. Solve $\begin{Bmatrix} r = 1 + \cos 2\theta \\ r = 1 - \sin 2\theta \end{Bmatrix}$ for $0° < \theta < 360°$.

By subtraction, $0 = \cos 2\theta + \sin 2\theta$. Hence the equations are satisfied if

$$\sin 2\theta = -\cos 2\theta,$$
or $$\tan 2\theta = -1.$$

Hence

$$2\,\theta = 135^\circ + n \cdot 360^\circ, \qquad \text{or } 2\,\theta = 315^\circ + n \cdot 360^\circ.$$

$$\theta = 62^\circ\,30'.0 + n \cdot 180^\circ, \qquad \theta = 157^\circ\,30'.0 + n \cdot 180^\circ.$$

n	0	1
θ	$62^\circ\,30'.0$	$242^\circ\,30'.0$
r	$1-\frac{1}{2}\sqrt{2}$	$1-\frac{1}{2}\sqrt{2}$

n	0	1
θ	$157^\circ\,30'.0$	$337^\circ\,30'.0$
r	$1+\frac{1}{2}\sqrt{2}$	$1+\frac{1}{2}\sqrt{2}$

44. Inverse trigonometric equations. Such equations may often be solved by reducing to equivalent equations involving direct trigonometric functions. The examples below illustrate methods of solving.

Example 1. For what values of x is

$$\text{arc tan}\,\frac{x}{2} - \text{arc tan}\,\frac{3}{5} = \frac{3\,\pi}{4},$$

if the inverse functions are limited to acute angles?

Let $\qquad A = \text{arc tan}\,\dfrac{x}{2} \quad \text{and} \quad B = \text{arc tan}\,\frac{3}{5}.$

Then $\qquad A - B = \dfrac{3\,\pi}{4} \quad \text{or} \quad A = B + \dfrac{3\,\pi}{4},$

and any trigonometric function of one member of the equation is equal to the same function of the other.
Hence

$$\tan A = \tan\left(B + \frac{3\,\pi}{4}\right) = \frac{\tan B + \tan\dfrac{3\,\pi}{4}}{1 - \tan B \tan\dfrac{3\,\pi}{4}}.$$

$$\frac{x}{2} = \frac{(\frac{3}{5}) + (-1)}{1 - (\frac{3}{5})(-1)} = \frac{-\frac{2}{5}}{\frac{8}{5}} = -\tfrac{1}{4}.$$

$$\therefore x = -\tfrac{1}{2}.$$

Example 2. For what value of x is arc sin x − arc cos $2x = 0$, considering only acute angles?

Let $\qquad A = \text{arc sin } x, \qquad B = \text{arc cos } 2x.$

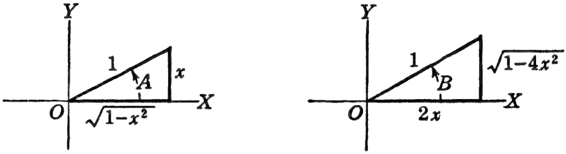


FIG. 51

Then

$$A - B = 0 \text{ or } A = B,$$

and any trigonometric function of A is equal to the same function of B.

$$\begin{aligned} \sin A &= \sin B, \\ x &= \sqrt{1 - 4x^2}, \\ x^2 &= 1 - 4x^2, \\ x &= \pm\tfrac{1}{5}\sqrt{5}. \end{aligned}$$

Checking to remove any extraneous roots introduced by squaring both sides of the equation gives $x = \frac{1}{5}\sqrt{5}$, or it might be noticed that x must be positive as the inverse functions are limited to acute angles.

EXERCISES

1. Example 1 of Art. 42 could also have been solved by taking $\dfrac{6}{3\sqrt{5}} = \cos\beta$. Solve that problem with this substitution.

2. Solve the equation of Example 2, Art. 43 by replacing (a) $\cos 2\theta$ by $\sin(90° + 2\theta)$; (b) $\sin 3\theta$ by $\cos(90° - 3\theta)$.

Find all values of x, or θ, from 0° to 360° that satisfy:

3. $2\sec^2 x - 5\tan x = 0$.

4. $\text{ctn } x = \sec x$.

5. $\sin 2x = \cos x$.

6. $\cos 2x = \sin x$.

7. $2\cos^2 x + 5\sin x - 4 = 0$.

8. $\sin 4x = \cos 2x$.

9. $\sec^2 2x + 3\tan 2x = 5$.

10. $\tan 2x \tan x = 1$.

11. $3 \sin x = 2 \cos^2 x$.

12. $2 \cos 2x + \cos x = 0$.

13. $2 \cos 4x + \cos 2x = 0$.

14. $\sin x \cos x = -\frac{1}{4}\sqrt{3}$.

15. $\cos \theta = \sqrt{3} \operatorname{covers} \theta$.

16. $3 \sin x - 4 \cos x = 1$.

17. $3 \sin x - 4 \cos x = 5$.

18. $4 \sin x - 3 \cos x = -2$.

19. $4 \sin x + 3 \cos x = -2$.

20. $\sin x - 2 \cos x = 1$.

21. $\sqrt{3} \sin x + \cos x = 2$.

22. $3 \sin x - \cos x = 1$.

23. $2 \sin x - 3 \cos x = -1$.

24. $3 \cos x - 2 \sin x = -1$.

25. $3 \sin x - 5 \cos x = 4$.

26. $3 \sin x - 5 \cos x = -4$.

27. $3 \cos x + 5 \sin x = 4$.

28. $3 \cos x + 2 \sin x = -3$.

29. $\cos 2x = \sin 3x$.

30. $\tan x = \operatorname{ctn} 2x$.

31. $\cos 2x = -\cos x$.

32. $\sin x = \cos 4x$.

33. $\sin x = \cos 3x$.

34. $\cos x = \sin 3x$.

35. $\tan 2x = \operatorname{ctn} 3x$.

36. $\operatorname{ctn} x = \tan 3x$.

37. $\cos \theta - \cos 3\theta = \sin 2\theta$.

38. $\sin 2x - 2 \sin x - 2 \cos x + 2 = 0$.

39. $\sin \theta - \sin 2\theta + \sin 3\theta = 0$.

40. $\sin \theta + \cos 2\theta - \sin 3\theta = 0$.

41. $\sin 5x - \sin 3x + \sin x = 0$.

42. $\cos (60° + x) - \cos (60° - x) = -\frac{1}{2}\sqrt{3}$.

43. $\sin (60° - x) - \sin (30° + x) = \frac{1}{2}\sqrt{2}$.

44. $\cos (50° + x) + \cos (50° - x) = 0.78543$.

45. $\sin (70° + x) + \sin (40° - x) = 0.99000$.

Solve the simultaneous equations:

46. $\begin{cases} r = a \sin \theta \\ r = a(1 - \sin \theta). \end{cases}$

47. $\begin{cases} r = a \sin \theta \\ r = a \cos 2\theta. \end{cases}$

48. $\begin{cases} r = a \\ r = 2a(1 + \cos \theta). \end{cases}$

49. $\begin{cases} r = 2 \cos \theta \\ r^2 = 4 \sin 2\theta. \end{cases}$

50. $\begin{cases} r = a \cos \theta \\ r = a \sin 3\theta. \end{cases}$

51. $\begin{cases} r = 2 + \cos \theta \\ r^2 = 4 \cos 2\theta. \end{cases}$

52. $\begin{cases} y = 1 + \cos 2x \\ y = 1 - \sin 2x. \end{cases}$

53. $\begin{cases} x^2 = 4 \sin 2y \\ x^2 \sin 2y = 1. \end{cases}$

54. $\begin{cases} r^2 = 3 \cos 2\theta \\ r^2 = 2 \cos^2 \theta. \end{cases}$

Considering only the acute angles represented by the inverse functions find the value of x:

55. $\tan^{-1} x = \tan^{-1} \frac{1}{4} + \tan^{-1} \frac{1}{13}$.

56. $\sin^{-1} \dfrac{4}{x} + \cos^{-1} \frac{1}{2} = \dfrac{\pi}{2}$.

57. $\tan^{-1} \frac{1}{2} - \tan^{-1} \frac{1}{7} = \arctan x$.

58. $2 \tan^{-1} \frac{2}{3} - \sin^{-1} \frac{8}{5} = \sin^{-1} \dfrac{x}{\sqrt{3}}$.

59. $x = 2 \tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{7}$.

60. $\tan^{-1} \dfrac{2m}{1 - m^2} = 2 \tan^{-1} x$.

61. $\tan^{-1} x - \operatorname{ctn}^{-1} x = \operatorname{ctn}^{-1} \frac{1}{2} - \tan^{-1} \frac{2}{3}$.

62. $x = \frac{1}{2} \operatorname{arc} \sin \frac{3}{5} + \operatorname{arc} \cos \frac{5}{13}$.

63. $x = \frac{1}{2} [\operatorname{arc} \sin \frac{4}{5} - \operatorname{arc} \sin \frac{8}{17}]$.

64. $\operatorname{arc} \tan \dfrac{x}{2} + \operatorname{arc} \operatorname{ctn} x = \operatorname{arc} \tan 2 + \operatorname{arc} \operatorname{ctn} 4$.

GENERAL EXERCISES

Given $A_3 = \tan^{-1} \frac{4}{3}$, $B_4 = \sec^{-1} \frac{17}{8}$, find the value:

1. $\sin 2 (A + B)$.

2. $\cos \left(\dfrac{A + B}{2}\right)$.

3. $\cos 2 (\pi - A)$.

4. $\tan \left(\dfrac{\pi}{2} + 2 A\right)$.

5. $\sin \left(\dfrac{\pi - A}{2}\right)$.

6. $\sin \left(\dfrac{3 \pi}{2} + 2 A\right)$.

7. $\sin (A + B) - \sin (A - B)$.

8. $\sin 2 \left(\dfrac{\pi}{4} - \dfrac{A}{4}\right)$.

9. If $\sin \dfrac{A}{2} = \dfrac{2}{7}$, $90° < \dfrac{A}{2} < 180°$, find the value of

(a) $\sin A$. (b) $\cos A$. (c) $\tan A$. (d) $\sin 2 A$.

Prove the following identities:

10. $\cos^4 2 x - \sin^4 2 x = \cos 4 x$

11. $\tan x + \operatorname{ctn} x = \dfrac{2}{\sin 2 x}$.

12. $\dfrac{\sin (x + 2 y) - \sin (x - 2 y)}{\sin y} = 4 \cos x \cos y$.

13. $\tan (45° + B) - \tan (45° - B) = 2 \tan 2 B$.

14. $(\operatorname{ctn} \theta - \tan \theta) \div (\operatorname{ctn} \theta + \tan \theta) = \cos 2 \theta$.

15. $\sin x = \dfrac{2}{\operatorname{ctn} \dfrac{x}{2} + \tan \dfrac{x}{2}}$.

16. $\cos 5 x \cos 2 x = \dfrac{\cos 7 x + \cos 3 x}{2}$.

17. $\dfrac{\sin 2 x + \sin x}{\cos 2 x + \cos x + 1} = \tan x$.

18. $(\sin\theta - \frac{1}{2}\sin 2\theta) \div (\sin\theta + \frac{1}{2}\sin 2\theta) = \tan^2\frac{\theta}{2}$.

19. $\left(\tan^2\frac{x}{2} + 1\right)\cdot\sin x = 2\tan\frac{x}{2}$.

20. $\dfrac{\sin 3x}{\sin x} - \dfrac{\cos 3x}{\cos x} = 2.$

21. $\dfrac{\cos^3\frac{x}{2} - \sin^3\frac{x}{2}}{\cos\frac{x}{2} - \sin\frac{x}{2}} = 1 + \frac{1}{2}\sin x.$

22. $\dfrac{\tan\frac{x}{2} + \operatorname{ctn}\frac{x}{2}}{\operatorname{ctn}\frac{x}{2} - \tan\frac{x}{2}} = \sec x.$

If A, B, and C are the angles of a triangle, show that:

23. $\tan A + \tan B + \tan C = \tan A \tan B \tan C.$

24. $\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$.

Prove, using only the smallest positive angle represented by each inverse function:

25. $\operatorname{arc\,sin}\dfrac{\sqrt{21}}{7} + \operatorname{arc\,tan}\dfrac{\sqrt{3}}{5} = 60°.$

26. $\operatorname{arc\,tan}\frac{1}{3} - \tan^{-1}(-1) = \operatorname{arc\,tan} 2.$

27. $\cos(2\operatorname{arc\,cos} x) = 2x^2 - 1.$

28. $\tan(\frac{1}{2}\operatorname{arc\,cos} x) = \sqrt{\dfrac{1-x}{1+x}}$.

29. If $y = \cos^{-1} m + \sin^{-1} n$, find $\cos y$.

30. Find x if $\sin\dfrac{A}{2}\cos\dfrac{B}{2} + \cos\dfrac{A}{2}\sin\dfrac{B}{2} = \sin\dfrac{A+B}{2}\cos\dfrac{x}{2}$.

31. Express as the cotangent of some angle $\dfrac{\sin\alpha - \sin\beta}{\cos\beta - \cos\alpha}$.

32. Find a and x if $\dfrac{\cos 3\theta}{\sin\theta} + \dfrac{\sin 3\theta}{\cos\theta} = a\operatorname{ctn} x.$

33. For what values of x is $\tan^{-1} h + \tan^{-1} k = \tan^{-1} x$?

34. Complete and prove, $\dfrac{\cos x - \cos 3x}{\sin 3x - \sin x} = ?\,(2x).$

35. Find x in terms of a and b if $\sin^{-1} x = 2 \cos^{-1} a + \cos^{-1} b$.

36. Find x in terms of a and b if $\tan^{-1} x = \cos^{-1} a + \frac{1}{2} \cos^{-1} b$.

37. Find the value of $\sin\left[\frac{\pi}{2} - 2 \operatorname{arc\,ctn} \sqrt{\frac{1+x}{1-x}}\right]$.

38. Show that $\operatorname{arc\,cos} \frac{2\sqrt{ab}}{a+b} = \operatorname{arc\,ctn} \frac{2\sqrt{ab}}{a-b}$ for an acute angle, and find the sine of the same angle. Is this always true?

39. If $\frac{3}{5} \sin C + \frac{4}{5} \cos C = \sin (C - B)$, in what quadrant must B lie? Find $\tan B$.

40. Is $\operatorname{arc\,cos} x - \operatorname{arc\,sin} x = \operatorname{arc\,cos} x\sqrt{3}$ an identity? If it is a conditional equation find one value of x that will satisfy it.

41. Show whether $1 - \sin^2 \phi = 3 \sin \phi \cos \phi$ is true for all values of ϕ; for any value of ϕ.

42. For what values of x is $\sin^{-1} x - \tan^{-1} \frac{\sqrt{1-x^2}}{x} = \frac{\pi}{2}$?

43. Solve for x in terms of n: $\tan^{-1} x + \tan^{-1} n = \operatorname{arc\,tan} (x - n)$.

44. Find a if $\operatorname{arc\,tan} a = \tan^{-1} (\sqrt{2} + 1) - \tan^{-1} (-\sqrt{2} - 1)$.

45. Solve for y: $\frac{\pi}{2} - 2 \operatorname{arc\,tan} y = \operatorname{arc\,ctn} 3y$.

46. Find x if $2 \operatorname{arc\,sin} x = \operatorname{arc\,tan} a$.

47. Determine a if $\operatorname{arc\,tan} a = 2 \operatorname{arc\,tan} \frac{1}{2} - \operatorname{arc\,tan} \frac{1}{7}$.

48. Find x if $\tan^{-1} x = \tan^{-1} \frac{1}{h-k} - \tan^{-1} \frac{1}{h}$.

49. Find x if $\operatorname{arc\,tan} (x+1) - \operatorname{arc\,tan}\left(\frac{1}{x-1}\right) = \operatorname{arc\,tan} 2$.

Solve for all angles from $0°$ *to* $360°$.

50. $\cos 4\theta = \cos 2\theta$.

51. $\frac{27 \sin x}{\operatorname{ctn} x} = \frac{8 \cos x}{\tan x}$.

52. $10 \cos x - 24 \sin x = 13$.

53. $\operatorname{ctn} y - \tan y = 2$.

54. $5 \sin \theta + \tan \theta = 0$.

55. $\sin x + \cos 2x = 1$.

56. $2 \sin x + 5 \cos x = 2$.

57. $6 \sin x + \cos x = 2$.

58. $\sec^2 \theta + \operatorname{ctn}^2 \theta = \frac{13}{3}$.

59. $\cos \theta = \sin 3\theta$.

60. $\cos 3x = \sin x$.

61. $\sin x - \sqrt{3} \cos x = 2$.

62. $3 \sin x + 4 \cos x = -5$.

63. $\frac{\sin (30° + x)}{\cos (60° + x)} = -2$.

64. $3 \sin x - 2 \cos x = -2$.

65. $\sin 4x = 2 \cos 2x$.

66. $\cos 2x = \sin 2x$.

67. $\cos 3\theta = -\sin 2\theta$.

68. $\cos 5x - \cos 3x + \sin x = 0$.

69. $\sin 5x + \cos 3x - \sin x = 0$.

70. $\sin 2x + \sin 4x = \sqrt{2} \cos x$.

71. $2 \sin \phi + 4 \cos \phi = -3$.

72. $\sin 2B - \sin B + 3 - 4\cos^2\frac{B}{2} = 0.$

73. $2\cos^2 2\theta + \cos 2\theta - 1 = 0.$

74. $\cos(105° - 7x) - \cos(75° - x) = \sin 4x.$

75. $\cos 2\theta + 2\sin^2\frac{\theta}{2} - 1 = 0.$

76. $\tan\left(\frac{\pi}{4} - \alpha\right) + \tan\left(\frac{\pi}{4} + \alpha\right) = 4.$

77. Two parallel chords in a circle of radius 8 in. are on the same side of the center and are 5 in. apart. One subtends twice as large a central angle as the other. Find the length of the shorter one.

78. In a circle of radius 5 in., two parallel chords are on opposite sides of the center and 7 in. apart. Find the length of their arcs if the larger chord subtends a central angle double that subtended by the smaller.

CHAPTER V

OBLIQUE TRIANGLES

45. Introduction. In engineering practice many problems arise which involve the solution of oblique triangles. These problems may be solved by the method of right triangles, but it is more convenient to employ formulas which express the relations between the sides of any triangle and the trigonometric functions of its angles. This chapter is devoted to the derivation of these laws and their applications.

46. The law of sines. *The sides of a triangle are proportional to the sines of the opposite angles.*

Consider the oblique triangle ABC, where a, b, c represent the lengths of the sides opposite the angles A, B, C, respectively. From the vertex C drop the perpendicular h upon the side AB (produced if necessary).

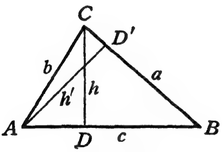


FIG. 52

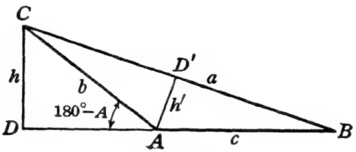


FIG. 53

In Fig. 52, In Fig. 53,

$$\frac{h}{b} = \sin A, \qquad \frac{h}{b} = \sin(180° - A) = \sin A, \quad (1)$$

$$\text{and} \quad \frac{h}{a} = \sin B. \qquad \text{and} \quad \frac{h}{a} = \sin B. \quad (2)$$

Hence, for either figure, dividing (1) by (2),

$$\frac{a}{b} = \frac{\sin A}{\sin B}, \quad \text{or} \quad \frac{a}{\sin A} = \frac{b}{\sin B}. \quad (3)$$

Similarly, by drawing the perpendicular h' from A to BC,

$$\frac{b}{c} = \frac{\sin B}{\sin C}, \quad \text{or} \quad \frac{b}{\sin B} = \frac{c}{\sin C}. \tag{4}$$

Combining (3) and (4),

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}. \qquad [27]$$

It should be observed that a triangle may be solved by means of the law of sines when two angles and a side, or when two sides and the angle opposite one of them are given.

EXERCISES

1. Derive, using Figs. 52 and 53: (a) $\dfrac{a}{\sin A} = \dfrac{c}{\sin C}$; (b) $\dfrac{b}{\sin B} = \dfrac{c}{\sin C}$.

2. Derive the law of sines by the use of formulas [10a, b, c] for the area of a triangle.

3. Derive: (a) $\dfrac{a}{\sin A} = 2R$; (b) $\dfrac{b}{\sin B} = 2R$; (c) $\dfrac{c}{\sin C} = 2R$; where $2R$ is equal to the diameter of the circle circumscribed about the triangle ABC.

SUGGESTION. Circumscribe a circle of radius R about the triangle ABC. Join the center O to the vertices A, B, and C. Drop the perpendicular OD upon BC. Since the central angle BOC and the inscribed angle A are both measured by the same arc, the angle $BOC = 2A$. From the right triangle BOD, (a) is easily obtained. Similarly for (b) and (c).

4. Solve for each part involved: (a) $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$; (b) $\dfrac{a}{\sin A} = \dfrac{c}{\sin C}$; (c) $\dfrac{b}{\sin B} = \dfrac{c}{\sin C}$.

5. Derive a formula for the area of a triangle when given: (a) a, A, B, C; (b) b, A, B, C; (c) c, A, B, C.

6. Prove that for any triangle: (a) $a = b\cos C + c\cos B$; (b) $b = a\cos C + c\cos A$; (c) $c = a\cos B + b\cos A$.

7. Prove that for any triangle: (a) $\dfrac{a-b}{c} = \dfrac{\sin \frac{1}{2}(A-B)}{\cos \frac{1}{2} C}$; (b) $\dfrac{c-b}{a} = \dfrac{\sin \frac{1}{2}(C-B)}{\cos \frac{1}{2} A}$; (c) $\dfrac{a-c}{b} = \dfrac{\sin \frac{1}{2}(A-C)}{\cos \frac{1}{2} B}$.

SUGGESTION. To prove (a), subtract both members of $\dfrac{b}{c} = \dfrac{\sin B}{\sin C}$ from $\dfrac{a}{c} = \dfrac{\sin A}{\sin C}$. Then, in the right-hand member of this equation, apply formula [24] to the numerator, formula [17] to the denominator, and simplify. Similarly for (b) and (c).

8. Prove that for any triangle: (a) $\dfrac{b+a}{c} = \dfrac{\cos \frac{1}{2}(B-A)}{\sin \frac{1}{2} C}$; (b) $\dfrac{c+b}{a} = \dfrac{\cos \frac{1}{2}(C-B)}{\sin \frac{1}{2} A}$; (c) $\dfrac{a+c}{b} = \dfrac{\cos \frac{1}{2}(A-C)}{\sin \frac{1}{2} B}$.

47. Applications of the law of sines. The following examples illustrate the use of the law of sines in solving triangles. These are grouped under two different cases and show the details of solution as well as a systematic arrangement.

The graphical solution has been used to detect large errors. If an accurate check is desired, one of Mollweide's equations, given in Exs. 7 and 8 of Art. 46, or the law of tangents (Art. 50), in a form involving as many of the computed parts as possible, should be used.

CASE I. *Given two angles and a side.*

Example 1. Solve the triangle when $b = 34.906$, $A = 98° 42' 43''$, $C = 31° 19' 11''$.

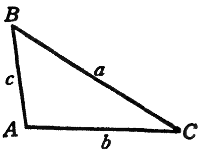


FIG. 54

Given	To find	Estimated
$b = 34.906$	$a = 45.062$	$a = 45$
$A = 98° 42' 43''$	$c = 23.697$	$c = 24$
$C = 31° 19' 11''$	$B = 49° 58' 6''$	$B = 50°$

$$B = 180° - (A + C) = 49° 58' 6''.$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} \quad \text{or} \quad a = \frac{b \sin A}{\sin B}. \qquad \frac{c}{\sin C} = \frac{b}{\sin B} \quad \text{or} \quad c = \frac{b \sin C}{\sin B}.$$

$\log a = \log b + \log \sin A - \log \sin B.$ $\log c = \log b + \log \sin C - \log \sin B.$

log b	1 54290
log sin A	9 99496−10
log (b sin A)	11 53786−10
log sin B	9 88405−10
log a	1 65381
a	45.062

log b	1.54290
log sin C	9.71585−10
log (b sin C)	11.25875−10
log sin B	9.88405−10
log c	1.37470
c	23 697

Example 2. Find the area of the triangle in Example 1.

$$K = \tfrac{1}{2} bc \sin A. \qquad [10a]$$

$$\log K = \log b + \log c + \log \sin A - \log 2.$$

log b	1 54290
log c	1.37470
log sin A	9 99496−10
log (bc sin A)	2.91256
log 2	0.30103
log K	2.61153
K	408.82

CASE II. *Given two sides and an angle opposite one of them.* This problem may have one solution, two solutions, or no solution. Hence the name — the **ambiguous case.**

Let b and c be the given sides and C the given angle. From the law of sines,

$$\sin B = \frac{b \sin C}{c}, \qquad (1)$$

and $$\log \sin B = \log b + \log \sin C - \log c. \qquad (2)$$

The ambiguity arises from the fact that from the above relations, B can have two values — if any at all — one acute and the other obtuse. The various possibilities are discussed on the left below and the corresponding graphical representations are shown on the right.

If the calculated $\log \sin B > 0$, $\sin B > 1$, which is impossible; hence no solution.

If the calculated $\log \sin B = 0$, $\sin B = 1$, $B = 90°$; hence a right triangle.

If the calculated $\log \sin B < 0$, $\sin B < 1$, two supplementary values of B are determined, one acute, the other obtuse; hence two solutions unless the obtuse value of B plus the given angle $C \gtreqless 180°$.

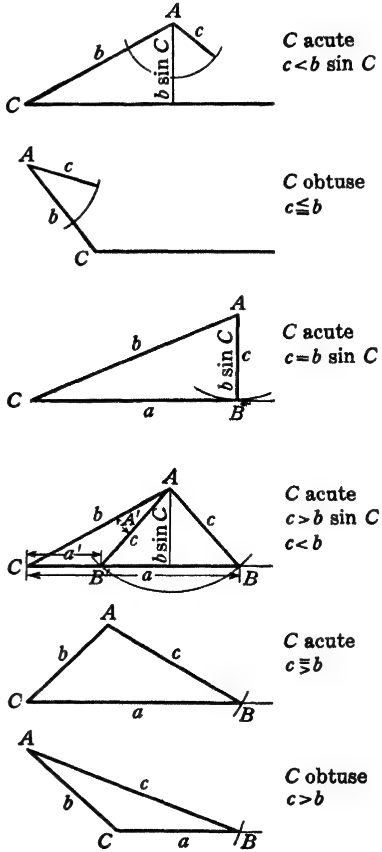


FIG. 55

Example 3. Solve the triangle when $a = 5.1297$, $c = 8.0064$, $A = 56° \, 9'.7$.

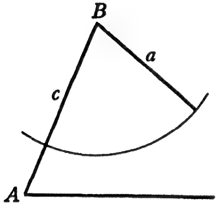


FIG. 56

Given	To find	Estimated
$a = 5.1297$	b No	No
$c = 8.0064$	B solution	solution
$A = 56° \, 9'.7$	C	

Here A is acute and $a < c$; hence there may be one solution, two solutions, or no solution.

$$\frac{a}{\sin A} = \frac{c}{\sin C} \quad \text{or} \quad \sin C = \frac{c \sin A}{a}.$$

$$\log \sin C = \log c + \log \sin A - \log a.$$

$\log c$	0 90344
$\log \sin A$	9 91940−10
$\log (c \sin A)$	0 82284
$\log a$	0 71009
$\log \sin C$	0 11275
C	Impossible

Example 4. Solve the triangle when $b = 9.2009$, $c = 12.794$, $B = 29° \, 34'.3$.

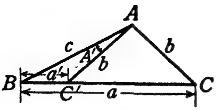


FIG. 57

Given	To find	Estimated
	Two solutions	
$b = 9.2009$	$a = 17.820$	$a = 18$
$c = 12.794$	$A = 107° \, 5'.8$	$A = 107°$
$B = 29° \, 34'.3$	$C = 43° \, 19'.9$	$C = 43°$
	$a' = 4.4346$	$a' = 4.5$
	$A' = 13° \, 45'.6$	$A' = 14°$
	$C' = 136° \, 40'.1$	$C' = 137°$

Here B is acute and $b < c$; hence there may be one solution, two solutions, or no solution.

$$\frac{c}{\sin C} = \frac{b}{\sin B} \quad \text{or} \quad \sin C = \frac{c \sin B}{b}.$$

$$\log \sin C = \log c + \log \sin B - \log b.$$

$\log c$	1.10701
$\log \sin B$	9.69330−10
$\log (c \sin B)$	10.80031−10
$\log b$	0 96384
$\log \sin C$	9.83647−10
C	43° 19′.9 or 136° 40′.1

NOTE. As previously mentioned, when $\log \sin C$ is given, C can have two values — one acute and the other obtuse. The obtuse value should always be found and a trial made to see if it leads to a second solution.

Since the obtuse value of C (136° 40′.1) plus the given angle B (29° 34′.3) is less than 180°, there are two solutions.

$C = 43° \ 19'.9.$

$A = 180° - (B + C) = 107° \ 5'.8.$

$$\frac{a}{\sin A} = \frac{b}{\sin B} \quad \text{or} \quad a = \frac{b \sin A}{\sin B}.$$

$$\log a = \log b + \log \sin A - \log \sin B.$$

$\log b$	0 96384
$\log \sin A$	9 98037−10
$\log (b \sin A)$	10 94421−10
$\log \sin B$	9 69330−10
$\log a$	1.25091
a	17.820

$C' = 136° \ 40'.1.$

$A' = 180° - (B + C') = 13° \ 45'.6.$

$$\frac{a'}{\sin A'} = \frac{b}{\sin B} \quad \text{or} \quad a' = \frac{b \sin A'}{\sin B}.$$

$$\log a' = \log b + \log \sin A' - \log \sin B.$$

$\log b$	0.96384
$\log \sin A'$	9.37631−10
$\log (b \sin A')$	10 34015−10
$\log \sin B$	9.69330−10
$\log a'$	0.64685
a'	4.4346

Example 5. Find the areas of the triangles in Example 4.

$K = \frac{1}{2}\, bc \sin A.$ $\qquad$ $K' = \frac{1}{2}\, bc \sin A'.$

$\log K = \log b + \text{lo6}\, c + \log \sin A - \log 2.$ $\quad$ $\log K' = \log b + \log c + \log \sin A' - 2.$

$\log b$	0 96384
$\log c$	1 10701
$\log \sin A$	9 98037−10
$\log\, (bc \sin A)$	2 05122
$\log 2$	0 30103
$\log K$	1 75019
K	56 259

$\log b$	0.96384
$\log c$	1.10701
$\log \sin A'$	9 37631−10
$\log\, (bc \sin A')$	1.44716
$\log 2$	0 30103
$\log K'$	1.14613
K'	14.000

EXERCISES

Solve the following triangles and also find the areas of the starred problems, having given:

1. $A = 47° 38'.2,\quad B = 69° 43'.7,\quad a = 28.073.$
2.* $A = 73° 7'.9,\quad C = 55° 28'.6,\quad b = 0.48431.$
3*. $b = 24.519,\quad c = 19.838,\quad B = 67° 32'.3.$
4. $a = 46.763,\quad c = 51.277,\quad A = 23° 51' 22''.$
5. $a = 4.2889, b = 2.0071,\quad B = 29° 52' 28''.$
6.* $B = 17° 18' 11'',\quad C = 61° 3' 42'',\quad c = 137.64.$
7. $A = 38° 44'.8,\quad B = 105° 18'.5,\quad b = 111.19.$
8.* $b = 493.44,\quad c = 369.14,\quad C = 35° 46'.5.$
9.* $a = 8.5792,\quad b = 9.7157,\quad A = 54° 41'.8.$
10. $a = 22001,\quad c = 20338,\quad C = 73° 30'.5.$
11. $B = 98° 51' 43'',\quad C = 47° 9' 21'',\quad b = 0.038002.$
12.* $B = 92° 12' 24'',\quad C = 64° 35' 49'',\quad a = 6.4817.$
13. $A = 63° 41'.2,\quad C = 69° 3'.9,\quad c = 129.44.$
14.* $a = 0.41008,\quad b = 0.29326,\quad A = 128° 23' 19''.$
15. $b = 0.43796,\quad c = 0.55768,\quad B = 41° 31'.6.$
16. $b = 1.5184,\quad c = 1.4765,\quad C = 76° 32' 21''.$
17*. $A = 28° 41'.1,\quad B = 12° 58'.2,\quad c = 85.858.$

18. $a = 0.049317$, $c = 0.082195$, $A = 36° 52'.2$.
19.* $A = 33° 49'.6$, $C = 80° 27'.7$, $a = 9.0010$.
20.* $a = 1111.1$, $b = 1271.5$, $B = 65° 56'.8$.
21. $a = 99.005$, $c = 121.67$, $C = 113° 8' 33''$.

Find the area of the triangle in:

22. Ex. 1. **24.** Ex. 7. **26.** Ex. 13. **28.** Ex. 18.
23. Ex. 4. **25.** Ex. 11. **27.** Ex. 15. **29.** Ex. 21.

48. The law of cosines. *In any triangle the square on any side is equal to the sum of the squares on the other two sides minus twice the product of these two sides into the cosine of the included angle.*

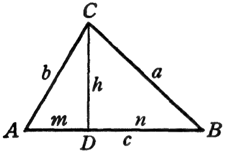


FIG. 58

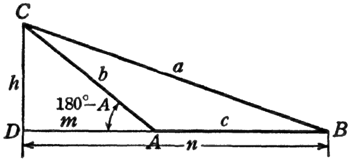


FIG. 59

In Fig. 58,

$$a^2 = h^2 + n^2$$
$$= h^2 + (c - m)^2$$
$$= h^2 + c^2 - 2\,cm + m^2$$
$$= (h^2 + m^2) + c^2 - 2\,cm.$$

But

$$h^2 + m^2 = b^2,$$

and

$$m = b \cos A.$$

In Fig. 59,

$$a^2 = h^2 + n^2 \qquad (1)$$
$$= h^2 + (m + c)^2 \qquad (2)$$
$$= h^2 + m^2 + 2\,cm + c^2 \qquad (3)$$
$$= (h^2 + m^2) + c^2 + 2\,cm. \qquad (4)$$

But

$$h^2 + m^2 = b^2, \qquad (5)$$

and

$$m = b \cos (180° - A)$$
$$= -b \cos A. \qquad (6)$$

Hence, for either figure,

$$\mathbf{a^2 = b^2 + c^2 - 2\,bc \cos A.} \qquad [28a]$$

Similarly,

$$b^2 = a^2 + c^2 - 2\,ac \cos B, \qquad [28b]$$
$$c^2 = a^2 + b^2 - 2\,ab \cos C. \qquad [28c]$$

Since these formulas are not adapted to logarithmic computation, they are useful only whenever the given sides are easily squared. With this restriction, the law of cosines should be employed in the solution of a triangle when two sides and the included angle, or when three sides are given.

EXERCISES

1. Derive, using Figs. 58 and 59: (a) [28b]; (b) [28c].

2. Derive, using the formulas in Ex. 5 of Art. 46: (a) [28a]; (b) [28b]; (c) [28c].

Suggestion. Multiply both members of the formulas (a), (b), and (c) of Ex. 5 by a, b, and c respectively. Then, add any two of the resulting equations and subtract the third.

3. Solve [28a, b, c] for the cosines of the angles in terms of the sides.

4. Solve: (a) [28a] for c; (b) [28b] for a; (c) [28c] for b.

49. Applications of the law of cosines.

Case III. *Given two sides and the included angle (given sides easily squared).*

Example 1. Given $a = 11$, $c = 15$, $B = 117° 49'.2$; find b.

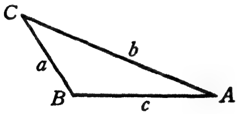


Fig. 60

Given	To find	Estimated
$a = 11$	$b = 22.361$	$b = 22$
$c = 15$		
$B = 117° 49'.2$		

$$b^2 = a^2 + c^2 - 2\,ac \cos B$$
$$= 121 + 225 - 330\,(-0.46669)$$
$$= 346 + 154.01 = 500.01.$$

$\log b^2$	2.69898
$\log b$	1.34949
b	22.361

If angles A and C were required, they could be found by the law of sines. A convenient check would be $A + B + C = 180°$, provided that the two angles were found independently.

CASE IV. *Given three sides (given sides easily squared).*

Example 2. Given $a = 5$, $b = 10$, $c = 12$; find the largest angle.

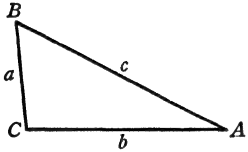


FIG. 61

Given	To find	Estimated
$a = 5$	$C = 100° 57'.2$	$C = 100°$
$b = 10$		
$c = 12$		

$$c^2 = a^2 + b^2 - 2ab \cos C.$$
$$144 = 25 + 100 - 100 \cos C.$$
$$\cos C = \frac{-19}{100} = -0.19000.$$
$$\text{arc cos } (0.19000) = 79° 2'.8.$$
$$\therefore C = 180° - 79° 2'.8 = 100° 57'.2.$$

Angles A and B could be found in a similar manner or by means of the law of sines. If the three angles were found independently, $A + B + C = 180°$ would be a convenient check.

EXERCISES

In the following triangles, find the indicated unknown, having given:

1. $a = 15$, $b = 23$, $C = 73° 29' 11''$; find c.
2. $b = 0.22$, $c = 0.33$, $A = 126° 52'.2$; find a.
3. $a = 1.3$, $b = 1.2$, $c = 1.5$; find the smallest angle.
4. $a = 125$, $b = 200$, $c = 250$; find the largest angle.
5. $a = 111$, $c = 93$, $B = 98° 56' 37''$; find b.
6. $a = 113$, $b = 15$, $c = 112$; find A.

Solve the following triangles, having given:

7. $a = 4$, $b = 2$, $c = 3$.
8. $a = 2.5$, $c = 3.8$, $B = 51° 44'.4$.
9. $a = 42$, $b = 35$, $C = 119° 38'.7$.
10. $a = 19$, $b = 28$, $c = 23$.
11. $a = 35$, $b = 12$, $c = 37$.

12. $b = 1250,\quad c = 1000,\quad A = 34°\,9'\,52''$.
13. $a = 0.7,\quad b = 0.3,\quad c = 0.5$.

Find the area of the triangle in:

14. Ex. 7. **15.** Ex. 12. **16.** Ex. 10. **17.** Ex. 9.

50. The law of tangents. *The sum of any two sides of a triangle is to their difference as the tangent of half the sum of the opposite angles is to the tangent of half their difference.*

From the law of sines,

$$\frac{a}{b} = \frac{\sin A}{\sin B}. \tag{1}$$

Adding unity to and subtracting unity from each member of (1), there results,

$$\frac{a}{b} + 1 = \frac{\sin A}{\sin B} + 1, \quad \text{or} \quad \frac{a+b}{b} = \frac{\sin A + \sin B}{\sin B}, \tag{2}$$

and

$$\frac{a}{b} - 1 = \frac{\sin A}{\sin B} - 1, \quad \text{or} \quad \frac{a-b}{b} = \frac{\sin A - \sin B}{\sin B}. \tag{3}$$

Dividing (2) by (3),

$$\frac{a+b}{a-b} = \frac{\sin A + \sin B}{\sin A - \sin B} \tag{4}$$

$$= \frac{2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)}{2\cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)} \qquad \text{Art. 40} \tag{5}$$

$$= \tan\tfrac{1}{2}(A+B)\operatorname{ctn}\tfrac{1}{2}(A-B). \tag{6}$$

$$\therefore \frac{\boldsymbol{a+b}}{\boldsymbol{a-b}} = \frac{\tan\frac{1}{2}(\boldsymbol{A+B})}{\tan\frac{1}{2}(\boldsymbol{A-B})}. \tag{29a}$$

Similarly,

$$\frac{\boldsymbol{a+c}}{\boldsymbol{a-c}} = \frac{\tan\frac{1}{2}(\boldsymbol{A+C})}{\tan\frac{1}{2}(\boldsymbol{A-C})}, \tag{29b}$$

and

$$\frac{\boldsymbol{b+c}}{\boldsymbol{b-c}} = \frac{\tan\frac{1}{2}(\boldsymbol{B+C})}{\tan\frac{1}{2}(\boldsymbol{B-C})}. \tag{29c}$$

The law of tangents is adapted to logarithmic computation and may be used in the solution of a triangle when two sides and the included angle are given. If in [29a], $b > a$, then $B > A$. Hence to avoid negative numbers, the formula should be written in the form

$$\frac{b+a}{b-a} = \frac{\tan \frac{1}{2}(B+A)}{\tan \frac{1}{2}(B-A)} \tag{7}$$

Similarly for [29b] and [29c].

EXERCISES

1. Derive: (a) [29b]; (b) [29c].
2. Give a geometrical proof of: (a) [29a]; (b) [29b]; (c) [29c].

Suggestion. To derive formula [29a], assume the given parts to be a, b, and C, with $a > b$. In the triangle ABC, draw the bisector of angle C meeting AB in D. Upon this bisector, produced if necessary, drop the perpendiculars AE and BF. Then, angle CAE = angle $FBC = \frac{1}{2}(A+B)$ and angle EAD = angle $FBD = \frac{1}{2}(A-B)$. From the similar right triangles ADE and FBD, $\tan \frac{1}{2}(A-B) = \frac{ED}{AE} = \frac{DF}{BF} = \frac{ED+DF}{AE+BF} = \frac{EC-FC}{AE+BF}$. Then substitute for each part in the last fraction employing the right triangles FBC and AEC and simplify. Similarly for formulas [29b] and [29c].

51. Application of the law of tangents.

Case III. *Given two sides and the included angle.*

If a, b, and C are the given parts, $\frac{1}{2}(A-B)$, assuming $a>b$, can be obtained from [29a], since $(a+b)$, $(a-b)$, and $\frac{1}{2}(A+B) = \frac{1}{2}(180° - C)$ are known. The addition and subtraction of the values of $\frac{1}{2}(A+B)$ and $\frac{1}{2}(A-B)$ give A and B respectively. The third side c can then be found by the law of sines. As a check, the law of sines, involving as many of the computed parts as possible, may be used.

Example. Solve the triangle when $a = 2.6983$, $c = 5.0021$, $B = 102°\ 18'\ 14''$ and also find the area.

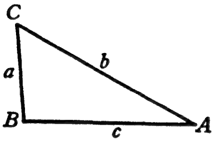


FIG. 62

Given	To find	Estimated
$a = 2.6983$	$A = 25°\ 18'\ 2''$	$A = 25°$
$c = 5.0021$	$C = 52°\ 23'\ 44''$	$C = 53°$
$B = 102°\ 18'\ 14''$	$b = 6.1690$	$b = 6.2$
	$K = 6.5936$	

$$c + a = 7.7004.$$

$$c - a = 2.3038.$$

$$C + A = 77°\ 41'\ 46''.$$

$$\tfrac{1}{2}(C + A) = 38°\ 50'\ 53''.$$

$$\frac{c+a}{c-a} = \frac{\tan\frac{1}{2}(C+A)}{\tan\frac{1}{2}(C-A)} \quad \text{or} \quad \tan\tfrac{1}{2}(C-A) = \frac{(c-a)\tan\frac{1}{2}(C+A)}{c+a}.$$

$$\log\tan\tfrac{1}{2}(C-A) = \log(c-a) + \log\tan\tfrac{1}{2}(C+A) - \log(c+a).$$

$\log(c - a)$	0.36244
$\log\tan\frac{1}{2}(C + A)$	9.90601−10
$\log[(c - a)\tan\frac{1}{2}(C + A)]$	10 26845−10
$\log(c + a)$	0.88651
$\log\tan\frac{1}{2}(C - A)$	9 38194−10
$\frac{1}{2}(C - A)$	13° 32′ 51″
$\frac{1}{2}(C + A)$	38° 50′ 53″
C	52° 23′ 44″
A	25° 18′ 2″

$$\frac{b}{\sin B} = \frac{a}{\sin A} \quad \text{or} \quad b = \frac{a\sin B}{\sin A}. \qquad K = \tfrac{1}{2}ac\sin B.$$

$$\log b = \log a + \log\sin B - \log\sin A. \quad \log K = \log a + \log c + \log\sin B - \log 2.$$

log a	0.43109
log sin B	9.98991−10
log (a sin B)	10.42100−10
log sin A	9.63080−10
log b	0 79020
b	6 1690

log a	0.43109
log c	0.69915
log sin B	9.98991−10
log (ac sin B)	11.12015−10
log 2	0 30103
log K	0.81912
K	6.5936

EXERCISES

Solve the following triangles and also find the areas of the starred problems, having given:

1. $a = 276.13$, $b = 199.86$, $C = 57° 34' 8''$.
2. $a = 0.026805$, $c = 0.049467$, $B = 35° 19'.2$.
3.* $b = 1111.4$, $c = 987.42$, $A = 11° 43' 27''$.
4. $a = 382.05$, $c = 294.77$, $B = 138° 21'.8$.
5. $a = 111.89$, $b = 163.51$, $C = 42° 19'.8$.
6.* $b = 0.54329$, $c = 0.74671$, $A = 26° 28'.2$.
7. $a = 0.29603$, $c = 0.10068$, $B = 64° 47' 14''$.
8. $b = 80.008$, $c = 99.205$, $A = 154° 3' 22''$.
9.* $a = 5.6682$, $c = 6.4399$, $B = 98° 10' 48''$.
10. $a = 86.094$, $b = 63.007$, $C = 121° 52'.7$.
11. $b = 2.1867$, $c = 1.9251$, $A = 144° 57'.3$.
12.* $a = 5.9809$, $b = 8.4035$, $C = 108° 22' 44''$.

Find the area of the triangle in:

13. Ex. 2. **15.** Ex. 8. **17.** Ex. 11.
14. Ex. 5. **16.** Ex. 10. **18.** Ex. 4.

52. The half-angle formulas in terms of the sides of a triangle. When the three sides of a triangle are given, the angles may be determined by the law of cosines. Thus, to find A,

$$\cos A = \frac{b^2 + c^2 - a^2}{2\,bc}. \tag{1}$$

This formula, however, is not adapted to logarithmic computation. A more convenient form is deduced as follows:

Upon substituting the value of $\cos A$ given in (1) in the formulas

$$\sin^2 \frac{A}{2} = \frac{1 - \cos A}{2}, \qquad [20] \qquad (2)$$

and

$$\cos^2 \frac{A}{2} = \frac{1 + \cos A}{2}, \qquad [21] \qquad (3)$$

there results,

$$\sin^2 \frac{A}{2} = \frac{1 - \dfrac{b^2+c^2-a^2}{2\,bc}}{2} \qquad \cos^2 \frac{A}{2} = \frac{1 + \dfrac{b^2+c^2-a^2}{2\,bc}}{2} \qquad (4)$$

$$= \frac{2\,bc - b^2 - c^2 + a^2}{4\,bc} \qquad = \frac{2\,bc + b^2 + c^2 - a^2}{4\,bc} \qquad (5)$$

$$= \frac{a^2 - (b^2 - 2\,bc + c^2)}{4\,bc} \qquad = \frac{(b^2 + 2\,bc + c^2) - a^2}{4\,bc} \qquad (6)$$

$$= \frac{a^2 - (b-c)^2}{4\,bc} \qquad = \frac{(b+c)^2 - a^2}{4\,bc} \qquad (7)$$

$$= \frac{(a+b-c)(a-b+c)}{4\,bc}. \qquad = \frac{(b+c+a)\,(b+c-a)}{4\,bc}. \qquad (8)$$

Let s denote the semi-perimeter of the triangle. Then, $2\,s = a+b+c$, $2\,s - 2\,a = -a+b+c$, $2\,s - 2\,b = a-b+c$, and $2\,s - 2\,c = a+b-c$.

Substituting these expressions in (8),

$$\sin^2 \frac{A}{2} = \frac{2\,(s-c) \cdot 2\,(s-b)}{4\,bc}, \qquad (9)$$

and

$$\cos^2 \frac{A}{2} = \frac{2\,s \cdot 2\,(s-a)}{4\,bc}. \qquad (10)$$

Whence,

$$\sin \frac{A}{2} = \sqrt{\frac{(s-b)\,(s-c)}{bc}}, \qquad [30a]$$

and

$$\cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}. \qquad [31a]$$

It should be noted that the sign before the radical is always positive, since one half of any angle of a triangle is an acute angle.

Similarly,

$$\sin\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{ac}}, \quad [30b]$$

$$\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}, \quad [30c]$$

$$\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ac}}, \quad [31b]$$

and

$$\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}. \quad [31c]$$

Formulas [30a, b, c] or [31a, b, c] may be used for logarithmic computation, but an even more convenient set of formulas is derived as follows:

$$\tan\frac{A}{2} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}. \quad (11)$$

Multiplying both numerator and denominator of the fraction under the radical of (11) by $(s - a)$, there results,

$$\tan\frac{A}{2} = \frac{1}{(s-a)}\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}. \quad (12)$$

Letting

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}, \quad (13)$$

$$\tan\frac{A}{2} = \frac{r}{s-a}. \quad [32a]$$

Similarly,

$$\tan\frac{B}{2} = \frac{r}{s-b}, \quad [32b]$$

and

$$\tan \frac{C}{2} = \frac{r}{s - c}. \qquad [32c]$$

Since the tangent varies more rapidly than either the sine or cosine, formulas [32a, b, c] give results which are usually more nearly accurate than those obtained from [30a, b, c] or [31a, b, c]. Hence [32a, b, c] are employed more often in the solution of a triangle when three sides are given.

EXERCISES

1. Derive: (a) [30b]; (b) [30c].
2. Derive: (a) [31b]; (b) [31c].
3. Derive: (a) [32b]; (b) [32c].

53. The area of a triangle expressed in terms of its sides. From Art. 33,

$$K(ABC) = \tfrac{1}{2} bc \sin A. \qquad (1)$$

Since

$$\sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2}, \quad [17] \qquad (2)$$

$$K(ABC) = bc \sin \frac{A}{2} \cos \frac{A}{2}. \qquad (3)$$

Substituting the values of $\sin \frac{A}{2}$ and $\cos \frac{A}{2}$ from [30a] and [31a],

$$K(ABC) = bc\sqrt{\frac{(s-b)(s-c)}{bc}} \cdot \sqrt{\frac{s(s-a)}{bc}}. \qquad (4)$$

Hence,

$$K = \sqrt{s(s-a)(s-b)(s-c)}, \qquad [33]$$

$$= s \cdot \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}, \qquad (5)$$

or

$$= rs. \qquad [34]$$

Formula [34] can also be obtained geometrically where r is the radius of the circle inscribed in a triangle whose sides are a, b, c. From Fig. 63,

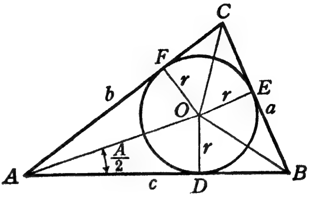


Fig. 63

$$
\begin{aligned}
K(ABC) &= K(AOB) + K(BOC) + K(COA) && (6)\\
&= \tfrac{1}{2}\,cr + \tfrac{1}{2}\,ar + \tfrac{1}{2}\,br = \tfrac{1}{2}\,r\,(a+b+c) && (7)\\
&= \tfrac{1}{2}\,r \cdot 2\,s = rs. && (8)
\end{aligned}
$$

Hence, a comparison of (8) with (5) shows that the **radius of the inscribed circle** is given by

$$r = \sqrt{\frac{(s-a)\,(s-b)\,(s-c)}{s}}. \qquad (9)$$

EXERCISE

1. Derive, using Fig. 63 and the value of r as given by (9): (a) [32a]; (b) [32b]; (c) [32c].

54. Application of the half-angle formulas.

Case IV. *Given three sides.*

When the three sides are given, formulas [32a, b, c] are used. Log r is obtained first, then the log-tangents of the three half-angles. A simple check is $\dfrac{A}{2} + \dfrac{B}{2} + \dfrac{C}{2} = 90°$.

Example 1. Solve the triangle when $a = 27.945$, $b = 41.189$, $c = 30.656$.

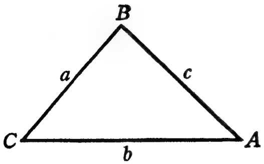


Fig. 64

Given	To find	Estimated
$a = 27.945$	$A = 42° \, 43'.2$	$A = 43°$
$b = 41.189$	$B = 89° \, 11'.4$	$B = 88°$
$c = 30.656$	$C = 48° \, 5'.4$	$C = 49°$

$$r = \sqrt{\frac{(s-a)\,(s-b)\,(s-c)}{s}}.$$

$$\log r = \tfrac{1}{2}\,[\log(s-a) + \log(s-b) + \log(s-c) - \log s].$$

$$\tan \frac{A}{2} = \frac{r}{s-a}.$$

$$\log \tan \frac{A}{2} = \log r - \log (s-a).$$

$$\tan \frac{B}{2} = \frac{r}{s-b}.$$

$$\log \tan \frac{B}{2} = \log r - \log (s-b).$$

$$\tan \frac{C}{2} = \frac{r}{s-c}.$$

$$\log \tan \frac{C}{2} = \log r - \log (s-c).$$

a b c	27 945 41.189 30 656
$2s$	99.790
s	49 895
$s-a$ $s-b$ $s-c$	21.950 8.706 19.239
Check: s	49 895

Check: $\frac{A}{2} + \frac{B}{2} + \frac{C}{2} = 90°.$

$\log (s-a)$	1.34143
$\log (s-b)$	0.93982
$\log (s-c)$	1.28419
log numerator	3.56544
$\log s$	1.69806
$\log r^2$	1 86738
$\log r$	0.93369
$\log \tan \frac{A}{2}$	9.59226−10
$\frac{A}{2}$	21° 21′.6
$\log \tan \frac{B}{2}$	9.99387−10
$\frac{B}{2}$	44° 35′.7
$\log \tan \frac{C}{2}$	9 64950−10
$\frac{C}{2}$	24° 2′.7

Example 2. Find the area of the triangle in Example 1.

$$K = rs.$$

$$\log K = \log r + \log s.$$

$\log r$	0.93369
$\log s$	1.69806
$\log K$	2.63175
K	428.30

If only the area of this triangle was desired, the value of log r would not be known. In this case, formula [33] would be more convenient.

EXERCISES

Solve the following triangles and also find the areas, having given:

1. $a = 62.409$, $b = 48.932$, $c = 87.795$.
2. $a = 9.8378$, $b = 7.0037$, $c = 8.8163$.
3. $a = 121.91$, $b = 135.39$, $c = 106.82$.
4. $a = 0.38199$, $b = 0.19005$, $c = 0.29848$.
5. $a = 534.37$, $b = 826.72$, $c = 555.34$.
6. $a = 0.014623$, $b = 0.019387$, $c = 0.024648$.
7. $a = 3.6845$, $b = 3.4983$, $c = 3.1326$.
8. $a = 76.943$, $b = 99.371$, $c = 61.176$.
9. $a = 1901.5$, $b = 1743.6$, $c = 2286.3$.

Find the area of the triangle in:

10. Ex. 2. 11. Ex. 5. 12. Ex. 9. 13. Ex. 4.

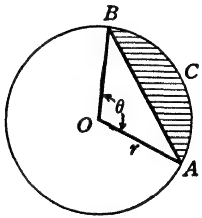


FIG. 65

55. Sector and segment areas of a circle. Let r be the radius of the circle and θ the number of radians in the central angle. Then,

$$\frac{\text{area of sector } OACB}{\text{area of circle}} = \frac{\theta \text{ radians}}{2\pi \text{ radians}}. \tag{1}$$

That is,

$$\frac{K(\text{sector } OACB)}{\pi r^2} = \frac{\theta}{2\pi}, \tag{2}$$

from which

$$K(\text{sector } OACB) = \frac{\theta}{2\pi} \cdot \pi r^2 = \tfrac{1}{2} r^2\theta. \tag{3}$$

Hence the formula,

$$\boldsymbol{K}(\textbf{sector}) = \tfrac{1}{2} \boldsymbol{r^2\theta}. \qquad [35]$$

The area of the segment ACB (shaded in Fig. 65) is given by the relation

$$K(\text{segment } ACB) = K(\text{sector } OACB) - K(\text{triangle } AOB) \quad (4)$$
$$= \tfrac{1}{2} r^2\theta - \tfrac{1}{2} r \cdot r \cdot \sin\theta. \quad (5)$$

Therefore,

$$K\ (\text{segment}) = \tfrac{1}{2} r^2(\theta - \sin\theta). \quad [36]$$

In using formulas [35] and [36], it is important to remember that θ must be expressed in radians.

EXERCISES

Find the areas of the following sectors and segments of a circle, having given:

1. $r = 14$ in., $\theta = 69° 14'.6$.
2. $r = 42$ cm., $\theta = 133° 37' 29''$.
3. $r = 1.3$ ft., $\theta = 200° 48'.3$.
4. $r = 35.6$ cm., $\theta = 317° 9' 55''$.
5. $r = 15$ in., $\theta = \frac{\pi}{8}$.
6. $r = 22$ in., $\theta = \frac{5\pi}{7}$.
7. $\theta = 43° 29' 45''$, intercepted arc $= 2\frac{1}{8}$ ft.
8. $\theta = \frac{7\pi}{6}$, intercepted arc $= 7\pi$ cm.
9. $\theta = 256° 51'.1$, intercepted arc $= 100.25$ cm.
10. $\theta = 6.2$, intercepted arc $= 18.6$ in.
11. $r = 26$ cm., subtended arc $= 39$ cm.
12. $r = 3.4$ in., subtended arc $= 13.6$ in.
13. Find the area of the larger segment of a circle bounded by a chord 25 in. long at a distance of 9 in. from the center.
14. The area of a sector is 120.45 sq. in. and its angle is $121° 19'.2$. Find the lengths of its radius and arc.
15. The area of a sector is 73.415 sq. cm. and its bounding arc is 14.683 cm. Find the angle at the center in degrees, minutes, and seconds.

16. A horizontal oil tank whose length is 30 ft. and radius 4 ft. is filled to a depth of 2 ft. How many cu. ft. of oil are in the tank?

17. A horizontal cylindrical tank, 20 ft. long and 6 ft. in diameter, is partly filled with water so that the greatest depth is 21 in. Find the number of gallons of water in the tank. (231 cu. in. = 1 gal.)

18. A cylindrical tank, axis horizontal, 15 ft. long and 5 ft. in diameter is filled with water so that the depth at the deepest point is 3 ft. Find the weight of the water if 1 cu. ft. weighs 62.5 lbs.

GENERAL EXERCISES

Solve the following triangles and also find the areas of the starred problems, having given:

1. $a = 7.0007$, $b = 4.6913$, $A = 111° 27' 37''$.
2.* $a = 4.8006$, $b = 3.1297$, $C = 107° 21'.8$.
3.* $a = 0.46193$, $b = 0.62987$, $c = 0.53722$.
4. $A = 51° 49'.2$, $B = 78° 9'.7$, $b = 33.003$.
5. $a = 13$, $b = 25$, $C = 109° 24'.7$.
6.* $A = 99° 12' 38''$, $C = 37° 51' 13''$, $a = 8.9076$.
7. $a = 7.0054$, $b = 3.9183$, $c = 5.3169$.
8.* $a = 13.289$, $c = 25.005$, $B = 64° 41' 32''$.
9. $a = 16$, $b = 22$, $c = 19$.
10.* $a = 22.585$, $b = 19.916$, $B = 53° 19'.4$.
11. $a = 109.24$, $c = 133.48$, $A = 61° 49' 9''$.
12.* $a = 115$, $c = 250$, $B = 64° 18' 29''$.
13.* $a = 88.093$, $b = 79.731$, $c = 93.846$.
14.* $a = 0.48231$, $c = 0.91007$, $C = 75° 11'.8$.
15. $b = 283.43$, $c = 399.89$, $A = 98° 57'.5$.
16. $a = 130$, $b = 100$, $c = 115$.
17. $b = 928.47$, $c = 1043.8$, $B = 62° 50'.3$.
18. $b = 0.094183$, $c = 0.048216$, $A = 35° 9' 14''$.
19.* $a = 3.4$, $b = 2.7$, $c = 4.5$.
20. $a = 600.37$, $b = 709.51$, $c = 840.58$.
21.* $B = 112° 32'.5$, $C = 41° 29'.2$, $c = 0.71045$.
22. $b = 6.2$, $c = 4.5$, $A = 32° 47'.4$.
23. $b = 0.040004$, $c = 0.050005$, $C = 38° 10' 24''$.

24. To find the distance from a point A to a point B on the opposite side of a river, a line AC and the angles CAB and ACB

were measured and found to be 428.53 ft., $64° 49'.5$, $51° 22'.9$ respectively. Find the distance AB.

25. Two points A and B are visible from a third point C, but not from each other. The distance AC, BC, and the angle ACB were measured and found to be 1930.8 ft., 1149.3 ft., and $47° 44'.8$ respectively. Find the distance AB.

26. Two inaccessible objects, A and B, are each viewed from two stations, C and D, on the same side of AB and 1941.6 ft. apart. The angles ACB, BCD, ADB, and ADC were measured and found to be $67° 28'.4$, $52° 19'.5$, $59° 36'.9$, and $41° 57'.8$ respectively. Find the distance AB.

27. Two vessels start from the same point and sail, one $S\ 38° 14' 27''\ W$ at the rate of 9 mi. per hr., and the other $N\ 7° 46' 19''\ W$ at the rate of 7 mi. per hr. How far apart will the ships be after 3 hrs.?

28. A horizontal oil tank, 25 ft. long and 6 ft. in diameter is filled with oil to a depth of 4 ft. Find the weight of the oil if 1 cu. ft. weighs 50 lbs.

29. From a tower 125 ft. high the angles of depression of two objects in the same horizontal plane as the foot of the tower are $27° 55'.9$ and $34° 12'.2$ and the horizontal angle subtended by the objects is $46° 30'.6$. Find the distance between the two objects.

30. Find the distance between the two objects in Ex. 29 if $46° 30'.6$ is the angle between the lines of sight instead of the horizontal angle subtended by the objects.

31. A grass plot in the form of a triangle has its sides 72.9 ft., 46.3 ft., and 81.7 ft. respectively. Find the area of the largest circular flower bed that can be made in the plot.

32. In the triangle ABC, a, b, and A are given. If $b = 74.621$ and $A = 29° 43'.8$, what values may a assume if the triangle has: (a) no solution?; (b) one solution?; (c) two solutions?

33. From a point 7 mi. from one end of a lake and 4 mi. from the other end, the lake subtends an angle of $69° 9' 49''$. Find the length of the lake.

34. Three intersecting streets form a plot of ground whose sides are 266.94 ft., 348.19 ft., and 312.77 ft. Find its area.

35. In a circle of radius 5 in., two parallel chords on opposite sides of the center are 7 in. apart. Find the area of the two segments

thus formed if the larger chord subtends an angle at the center double that subtended by the smaller chord.

36. From the ridge of a mountain range the angles of depression of the sides are $48° 41'.7$, $54° 23'.2$ respectively, and the corresponding distances from the ridge to the ends of a tunnel below (not horizontal) are 3714 ft. and 4157 ft. Find the length of the tunnel.

37. The diagonals of a parallelogram are 175.36 cm. and 212.73 cm. long and meet in an angle of $41° 37'.2$. Find the area of the parallelogram.

38. Two sides of a parallelogram are 12.095 in. and 15.162 in. long and the angle between them is $56° 59'.9$. Find the lengths of the diagonals.

39. One side of a triangle is 14.896 cm. longer than the other, and the angles opposite are $19° 16'.1$ and $53° 7'.8$. Solve the triangle and also find the area.

40. In surveying a field, a thick wood prevents the measurement of the angle ABD and the distance BD. A fourth point C was then located on the same side of AB as D and the distances BC, CD, and the angles ABC and BCD were measured. They were found to be 734.78 ft., 891.28 ft., $68° 36' 14''$, and $57° 13' 38''$ respectively. Find the angle ABD and the distance BD.

41. Find the area of the segment of a circle of radius 16 in. bounded by an arc of 72 in.

42. Two sides of a triangle are 13.462 and 20.005, and the difference between the angles opposite these sides is $21° 9'.1$. Solve the triangle.

43. A tower is situated on a hill which inclines at an angle of $17° 41'.7$ to the horizontal. The angle of elevation of the top of the tower from a point on the hill was measured and found to be $53° 18'.7$. At a point 100 ft. farther down the hill and in the same vertical plane as the tower, the angle of elevation was found to be $41° 28'.1$. Find the height of the tower.

44. In the triangle ABC, $a = 222.76$, $b = 444.38$, and $C = 17° 17'.6$. Find the length of the altitude from C to AB.

45. Find the difference of the areas of the two triangles determined by $b = 21.465$, $c = 16.009$, $C = 31° 52'.2$ without solving for the area of either of the given triangles.

46. A horizontal cylindrical tank, 7 ft. in diameter and 25 ft. long, is partly filled with water so that the wetted arc of the tank is 7.7 ft. How many gallons of water are there in the tank allowing $7\frac{1}{2}$ gallons to the cubic foot?

47. Two trees on a horizontal plane are 200 ft. apart. At their bases the angular elevation of one is double that of the other; but halfway between them, the elevations are complementary. Find the difference in the heights of the trees.

48. A flagpole makes an angle of $108° \ 17'.9$ with the inclined plane on which it stands; and at a distance of 92.623 ft. from its base, measured down the plane, the angle subtended by the flagpole is $25° \ 23'.6$. Find the height of the flagpole.

49. The angles of depression of the ends of a lake from the top of a hill 212 ft. above the level of the lake were measured as $9° \ 26' \ 18''$ and $15° \ 55' \ 43''$. The angle between the two lines of sight was $46° \ 27'.6$. Find the length of the lake.

50. Two parallel chords in a circle of radius 8 in. are on the same side of the center and are 5 in. apart. One subtends twice as large a central angle as the other. Find the area within the circle bounded by the two chords.

51. An observation tower 183.45 ft. high is situated at the top of a hill; 555 ft. down the hill the angle between the surface of the hill and a line to the top of the tower is $9° \ 56' \ 47''$. Find the distance to the top of the tower and the inclination of the surface of the hill to a horizontal plane.

52. At a certain point the length of a pond subtends an angle of $142° \ 27'.5$, and the distances from the point to the two extremities of the pond are 85 yds. and 55 yds. respectively. Find the length of the pond.

53. From the top of a lighthouse 250 ft. above sea level, the angle of depression of a ship was $9° \ 14'.9$; two minutes later it was $12° \ 42'.8$. Assuming the ship to be travelling in a straight course, find the distance it travelled if the horizontal angle between the two directions of the ship at the two instants was $131° \ 22'.4$.

54. The perimeter of a triangle is 196. The angle at C is double that at A, and the angle at A double that at B. Find the sides of the triangle.

55. Find the area of the circle (a) circumscribed by and (b) inscribed in the triangle whose sides are 72.963, 46.305 and 81.722.

56. A flagstaff 120 ft. high stands on the face of a hill whose inclination to the horizon is 28° 43′ 9″. At a point up the hill from the flagstaff, the angle of depression of its top is 21° 2′ 58″. Find the distance of the observer from the top of the flagstaff.

57. Two stations B and C are situated on a horizontal plane 1200 ft. apart. At B the angle of elevation of an aeroplane, which is directly above a point A in the same plane as B and C, is 61° 29′ 35″ and the horizontal angle at B subtended by A and C is 53° 11′ 51″, while at C the horizontal angle subtended by B and A is 71° 36′ 41″. Find the height of the aeroplane.

58. The area of a triangle is 1267.7 sq. ft. If $C = 68° 40'.5$ and $a = 89.478$ ft., solve the triangle.

59. To find the distance from A to an inaccessible point D, a straight base line ABC was located and the following measurements were recorded: $AB = 200.00$ ft., $BC = 150.00$ ft., $ABD = 111° 28'.5$, and $BCD = 68° 33'.2$. Find the distance from A to D.

60. A cliff 375 ft. high is observed to be due south of a ship and at an elevation of 28° 12′ 49″. After sailing a distance S 42° 38′ 57″ W the angle of elevation was found to be 37° 7′ 13″. How far did the ship sail?

61. A surveyor running a line due east from a point B, encounters a swamp at C. In order to continue the line beyond the swamp, he changes his direction at C to S 47° 00′ 00″ E for 2500.0 ft. to D, then turns to N 52° 00′ 00″ E. How far should he continue on this course to reach a point E on the continuation of BC?

62. A church is at the top of a straight street having an inclination of 12° 27′.9 to the horizontal. A straight line 105¼ ft. long is measured along the street in the direction of the church and at its extremities the angles of elevation of the church are 41° 42′.6 and 59° 7′.5. Find the height of the church.

63. From the top of a mountain the angle of depression of an object N 21° 12′ 40″ W in the horizontal plane below is 45°, while the depression angle of another object N 8° 47′ 20″ E in the same plane is 30°. Show that the distance between the objects is equal to the height of the mountain.

64. An observation tower 175 ft. high has an elevation of $27° 49'.9$ from A and $33° 7'.7$ from B, which is 225 ft. away from A and such that A, B, and the foot of the tower are in the same horizontal plane but not in the same straight line. Find the horizontal angle at the foot of the tower subtended by the line AB.

65. A tug that can steam 21 mi. per hr. is at a point B. It wishes to intercept a steamer as soon as possible that is due east at a point C and making 17 mi. per hr. in a direction $N\ 21° 0'.0\ W$. Find the direction the tug must take and the time it will take if C is 2.0671 mi. from B.

CHAPTER VI

GRAPHICAL REPRESENTATION OF TRIGONOMETRIC FUNCTIONS

56. Introduction. In this chapter it will be shown how to represent graphically some quantities expressed by trigonometric functions, especially by sines, cosines, or tangents. Such a graphical representation exhibits very clearly such general properties of the trigonometric functions as variation and periodicity. The compounding of such curves, especially of sines and cosines, will also be discussed. These graphs will also be found useful in finding approximate solutions of equations involving trigonometric functions of one unknown angle. Several of the problems in the general exercises at the end of this chapter afford good illustrations of practical problems of this nature.

57. Graphical representation of the sine, cosine and tangent. Such a representation may be effected by locating points, using the different values of the angle as abscissas and the corresponding function values as ordinates, then drawing a smooth curve through these points taken in order of increasing angles. In making a table of values, the angles are chosen first and the values of the abscissa, x, that will make the angle 0, $\frac{\pi}{2}$, π, and other multiples of $\frac{\pi}{2}$ are especially important. Multiples of $\frac{\pi}{6}$, $\frac{\pi}{4}$, and $\frac{\pi}{3}$ are the values of the angle ordinarily used for intermediate points, especially since the functions of these angles have been developed in Art. 7. After drawing a few of these graphs the student will see why it is better to choose these angles in place of such units as 1, 2, and 3. While an advantage to choose these multiples and fractional parts of π for the angle,

the student may find it easier, especially when using coordinate paper, to change these to the decimal system before plotting. This change is easily made by using Table I, or by slide-rule.

In the tables of values, derived for the problems below, the order in which the table should be started is indicated by the symbols ① and ②, the order of filling out the other parts being not so important. Practice alone will give judgment as to how far apart the angles should be chosen. To take an angle every $\frac{\pi}{6}$ is a good rule for beginners. The unit of angular measure is the radian, this choice of unit being necessary in many of the operations of calculus and other branches of mathematics. The length of the unit on each axis should be the same unless special circumstances make modification desirable. In the graphs of this article the x-axis may have a scale indicated both in terms of π radians and unit radians. After practice the student should be able to choose the one of these two scales that best fits his purpose.

The manner of plotting will also be shown by examples.

Example 1. Graph the sine curve, $y = \sin x$, to show one period.

The values for the angles will be taken at distances of $\frac{\pi}{6}$ apart, and x must vary through 2π to give a period as has been shown in Art. 29. It is usually desirable to graph the curve where it crosses the y-axis and to have the major portion to the right of the y-axis.

x ①	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π	$\frac{13\pi}{6}$
x	0	0.53	1.05	1.57	2.09	2.62	3.14	3.67	4.19	4.71	5.24	5.76	6.28	6.81
$\sin x$ ②	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{-\sqrt{3}}{2}$	-1	$\frac{-\sqrt{3}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$
$y=\sin x$	0	0.50	0.87	1.00	0.87	0.50	0	−0.50	−0.87	−1.00	−0.87	−0.50	0	0.50

In the graph below one period of the curve has been plotted, each point plotted being shown by the symbol O. The curve may be extended in either direction from knowl-

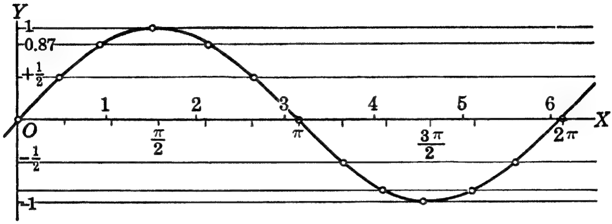


FIG. 66. — $y = \sin x$

edge of its period. As the values $y = 0.50$, $y = 0.87$, $y = 1.00$ and their corresponding negative values occur regularly, it is an advantage, especially on unruled paper, to draw as guide lines, the lines $y = 1$, $y = 0.50$, etc.

Example 2. Graph the tangent curve, $y = \tan x$, from $x = -\frac{\pi}{2}$ to $x = \frac{5\pi}{2}$.

The table of values may be found as indicated in Example 1. Since $\tan x$ changes rapidly as x varies from $\frac{\pi}{3}$ to $\frac{\pi}{2}$, the tables of natural values have been used to find the point at which $x = \frac{5\pi}{12}$. After plotting the points found, the curve has been extended by use of symmetry and the lines along which $y = .58$, 1.73, $-.58$, and -1.73. The points found in this way are indicated by the symbol ⊗.

x ①	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$	$\frac{7\pi}{12}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{17\pi}{12}$	$\frac{3\pi}{2}$
x	0	0.53	1.05	1.26	1.57	1 89	2 09	2 62	3.14	3.67	4.19	4.45	4.71
$\tan x$ ②	0	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$	3.73	∞	-3 73	$-\sqrt{3}$	$\frac{-\sqrt{3}}{3}$	0	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$	3.73	∞
$y = \tan x$	0	0.58	1.73	3.73	∞	-3 73	-1.73	-0 58	0	0.58	1.73	3.73	∞

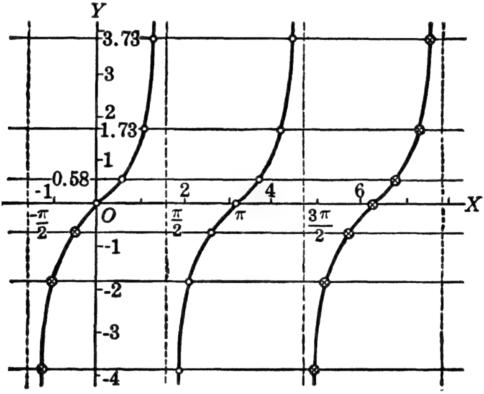


FIG. 67. — $y = \tan x$

Example 3. Graph the curve $y = 2\sin\frac{2x}{3}$ through one period.

In graphing such a curve, the same values are chosen for $\frac{2x}{3}$ as were chosen for x in graphing $y = \sin x$; in fact in any curve those values for the angle will give the most important points on the curve. After choosing these values for $\frac{2x}{3}$, the corresponding values for x and y may be found. Thus when $\frac{2x}{3} = \frac{\pi}{3}$, $x = \frac{3}{2}\cdot\frac{\pi}{3} = \frac{\pi}{2}$ and $y = 2\sin\frac{\pi}{3} = 2\cdot\frac{1}{2} = 1$. The corresponding values of x and y

are also put in decimal form. It is to be remembered that the values to be plotted are the values of x and the corresponding values of y.

$\frac{2x}{3}$ ①	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π	$\frac{9\pi}{4}$	$\frac{5\pi}{2}$	$\frac{11\pi}{4}$	3π
x	0	0 79	1 57	2 36	3 14	4 28	4 71	5.50	6 28	7 07	7 85	8.64	9 42
$\sin\frac{2x}{3}$ ②	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{-\sqrt{3}}{2}$	-1	$\frac{-\sqrt{3}}{2}$	$-\frac{1}{2}$	0
$\sin\frac{2x}{3}$	0	0 50	0 87	1 00	0 87	0 50	0	−0 50	−0 87	−1 00	−0 87	−0 50	0
$y = 2\sin\frac{2x}{3}$	0	1 00	1.74	2 00	1 74	1 00	0	−1 00	−1 74	−2 00	−1.74	−1 00	0

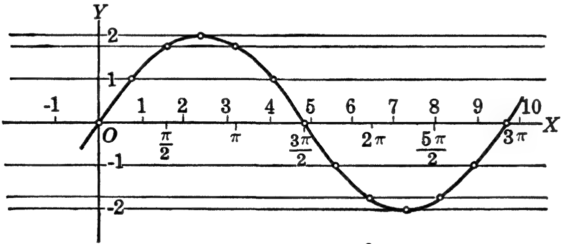


Fig. 68. — $y = 2\sin\frac{2x}{3}$

As can be seen from the graph or from the table of values, the values of y lie between $+2$ and -2 and vary from one to the other repeatedly. Under such conditions the curve is said to have an **amplitude** of 2. Its period is 3π.

Example 4. Graph $y = \frac{1}{2}\cos\frac{\pi x}{2}$ from -1 to 5.

Following the procedure of Example 3, values are first chosen for

$\frac{\pi x}{2}$, and the values 0, $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$ are especially important. Only one intermediate value between each of these has been used.

$\frac{\pi x}{2}$	①	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
x		0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4
$\cos\frac{\pi x}{2}$	②	1	$\frac{\sqrt{2}}{2}$	0	$-\frac{\sqrt{2}}{2}$	-1	$-\frac{\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	1
$\cos\frac{\pi x}{2}$		1	0.71	0	−0.71	−1	−0 71	0	0 .71	1
y		0 5	0.35	0	−0.35	−0 5	−0.35	0	0.35	0.50

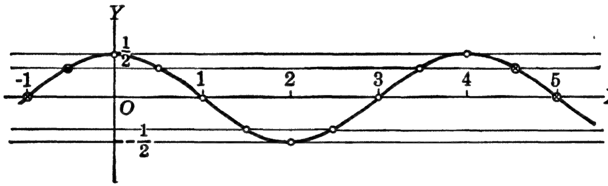


FIG. 69. — $y = \frac{1}{2}\cos\frac{\pi x}{2}$

This curve has a period of 4 and an amplitude of $\frac{1}{2}$.

Example 5. Graph $y = 2\sin(3x - 2)$ from $x = -1$ to $x = 5\frac{1}{2}$. A table of values will be necessary for one period, then the curve may be extended by symmetry to the required limits. The values for $3x - 2$ are first chosen, and it will be easier to reduce these to decimal form before finding the corresponding values for x. The point where the curve crosses the y-axis has been determined in addition to the usual values. The accuracy of the graph may be checked by determining the value of y that corresponds to some value of x not given in the table. The same value of y, approximately, should appear on the graph. The point where the curve crosses the y-axis, if not plotted, is a good point at which to check.

$3x-2$ (1)	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π	
$3x-2$	0	0 53	1 05	1 57	2 09	2 62	3 14	3.67	4.19	4.71	5 24	5.76	6.28	−2.0
x	0.67	0 84	1 02	1.19	1 36	1 54	1.72	1.89	2.06	2.24	2.41	2 59	2.76	0
$\sin(3x-2)$ (2)	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{-\sqrt{3}}{2}$	−1	$\frac{-\sqrt{3}}{2}$	$-\frac{1}{2}$	0	−0.91
y	0	1 00	1 74	2 00	1.74	1 00	0	−1 00	−1 74	−2 00	−1.74	−1.00	0	−1.82

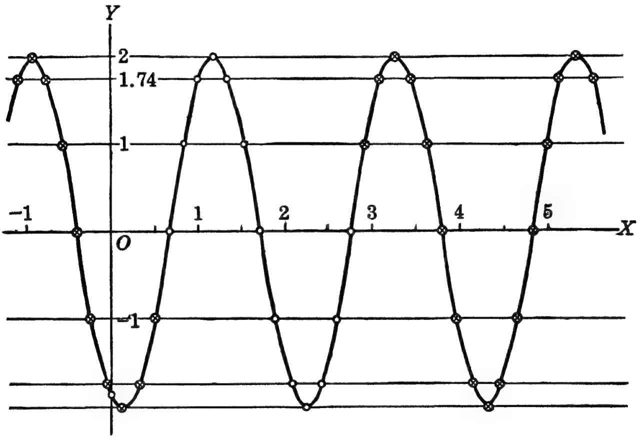


FIG. 70. — $y = 2\sin(3x - 2)$

Example 6. Graph $y = 2\sin x + \frac{1}{2}\sin 2x$ from $x = 0$ to $x = 2\pi$.

This graph is made by adding the ordinates of two curves, $y = 2\sin x$ and $y = \frac{1}{2}\sin 2x$, called **auxiliary** or **component** curves. This is known as **compounding** curves graphically. After graphing several sine and cosine curves, the student should be able to draw such component curves from their high and low points and intercepts on the x-axis with sufficient accuracy for this problem.

$y = 2 \sin x$

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin x$	0	1	0	-1	0
y	0	2	0	-2	0

$y = \frac{1}{2} \sin 2x$

$2x$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
$\sin 2x$	0	1	0	-1	0
y	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	0

To add the ordinates, lines are drawn parallel to the y-axis at irregular intervals as shown. At $x = OM$, the ordinates of the two curves are MP_1 and MP_2 respectively. Taking $P_2P = MP_1$ gives a line $MP = MP_2 + MP_1$. Hence P is a point on the required curve. At $x = OA$, the ordinates are AB and AC. By taking D so that $CD = AB$, $AD = AC + CD = AC + AB$. Hence D is a point on the required curve. The addition may be done by using dividers or if the graphing is done on rectangular coördinate paper, the student may find it more convenient to use the numerical value for each ordinate. Thus $AB + AC = 0.3 + (-1.8) = -1.5$, which is the length of AD.

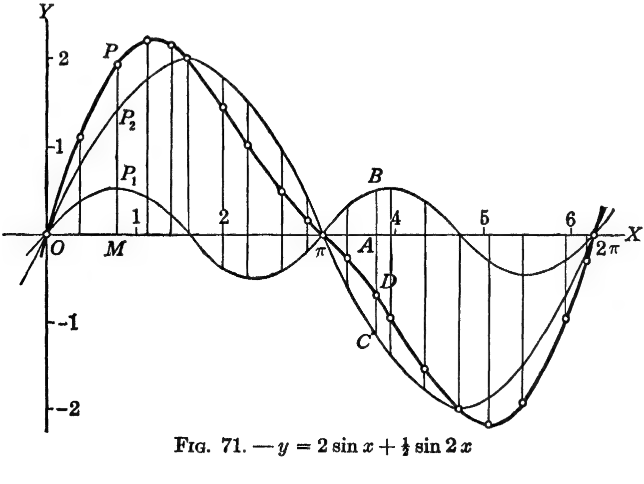


FIG. 71. — $y = 2 \sin x + \frac{1}{2} \sin 2x$

The period of $y = 2 \sin x + \frac{1}{2} \sin 2x$ can be seen from the graph to be 2π. Its amplitude cannot be determined from the graph as there is no assurance that the high point has been exactly located.

The curve $y = 2 \sin x + \frac{1}{2} \sin 2x$ could have been graphed directly from a table of values. It is, however, difficult to determine what values of x to choose. This method of drawing a graph from component graphs has even wider applications than are indicated in this book.

The curve $y = 2 \sin x - \frac{1}{2} \sin 2x$ can also be graphed from the same auxiliary or component curves by the subtraction of ordinates. To subtract, plot the negative of the subtrahend and add to the minuend. This is equivalent to graphing $y = 2 \sin x$ and $y = -\frac{1}{2} \sin 2x$ and adding the ordinates.

The amplitude of curves obtained by addition of ordinates are not definitely shown by the graph. The period of such a curve is the least common multiple of the periods of the component curves. Where the periods of the component curves are incommensurable, as π and 4, the resulting curve has no period and the function is called a non-periodic function.

EXERCISES

Graph the following curves between the limits indicated and give the period and amplitude of each where they are given by the graph.

1. $y = \cos x$, $-\pi$ to 2π.
2. $y = 3 \sin 2x$, $-\pi$ to π.
3. $y = \frac{1}{2} \cos 3x$, 0 to π.
4. $y = \frac{1}{2} \tan \frac{x}{2}$, 0 to π.
5. $y = 2 \tan 2x$, $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.
6. $y = \frac{1}{2} \tan \frac{\pi x}{2}$, -2 to 4.
7. $y = 2 \sin \frac{\pi x}{3}$, -3 to 6.
8. $y = 3 \cos \frac{\pi x}{4}$, one period.
9. $y = 2 \sin \left(x - \frac{\pi}{4}\right)$, 0 to 3π.
10. $y = 2 \sin (x + 2)$, -2 to 2π.

11. $y=\frac{1}{2}\cos(x+2)$, $\frac{-\pi}{2}$ to $\frac{3\pi}{2}$. **12.** $y=\tan\left(x-\frac{\pi}{4}\right)$, $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

13. $y=2\sin x+\frac{1}{3}\sin 3x$, 0 to 3π.

14. $y=\sin x+\frac{1}{3}\sin\frac{3x}{2}$, 0 to 2π.

15. $y=2\sin\frac{x}{2}+3\cos\frac{x}{4}$, $-\pi$ to 8π.

16. $y=2\sin x+\frac{1}{2}\sin 4x$, one period.

17. $y=2\sin\frac{x}{2}+6\cos\frac{x}{4}$, 0 to 8π.

18. $y=2\cos x+\frac{1}{2}\sin\frac{\pi x}{3}$, $-\pi$ to 2π.

19. $y=2\sin x-\frac{1}{2}\cos\frac{\pi x}{2}$, -2 to 2π.

20. $y=x+\sin x$, -2 to 3.

21. $y=\cos x+\frac{x}{2}$, -1 to 3.

22. $y=\frac{x^2}{16}-2\cos\frac{\pi x}{2}$, -4 to 4.

23. $y=\sqrt{16-x^2}+\sin\pi x$, -2 to 6.

24. Graph on the same axes and with the same scale, $y=\sin\left(x+\frac{\pi}{2}\right)$ and $y=\cos x$. Account for the relation between these curves.

25. Determine a value for b such that the graph of $y=\sin\left(x+\frac{\pi}{2}\right)$ will coincide with $y=\sin(x+b)$.

26. Do the graphs of $y=\sin 2x$ and $y=\sin(x+b)$ coincide for any values of b? If any, find them.

58. Graphical representation of inverse trigonometric functions. To obtain the graph of an equation involving an inverse trigonometric function, the equation is first changed into one involving the direct form of the trigonometric function. Then the procedure is the same as that of the preceding article.

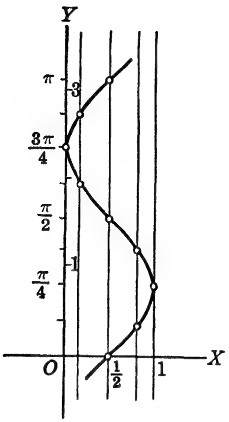


FIG. 72.—$y = \frac{1}{2}$ arc sin $(2x - 1)$

Example 1. Graph

$$y = \tfrac{1}{2} \text{ arc sin } (2x - 1)$$

from

$$y = 0 \text{ to } y = \pi.$$

To change to the direct form, the given equation

$$y = \tfrac{1}{2} \text{ arc sin } (2x - 1)$$

is first written in the form

$$2y = \text{arc sin } (2x - 1),$$

and then

$$2x - 1 = \sin 2y.$$

In making a table for graphing, values for $2y$ are first chosen. The complete tabulation of values is shown in the following table.

$2y$ ①	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
y	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$	$\frac{5\pi}{8}$	$\frac{3\pi}{4}$	$\frac{7\pi}{8}$	π
$\sin (2y)$ ②	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{2}}{2}$	0	$-\frac{\sqrt{2}}{2}$	-1	$-\frac{\sqrt{2}}{2}$	0
$2x - 1$	0	0.71	1	0.71	0	-0.71	-1.00	-0.71	0
x	0.50	0.86	1	0.86	0.50	0.15	0	0.15	0.50

EXERCISES

Graph the following showing at least one period:

1. $y = \frac{2}{3}$ arc cos $3x$.
2. $3y = 2$ arc sin $\frac{x}{2}$.
3. $y = \frac{1}{2}$ arc tan $2x$.
4. $x = \frac{1}{2}$ arc tan $2y$.
5. $2y =$ arc cos $(x - 2)$.
6. $2y =$ arc cos $(2x - 1)$.

7. $2y = \text{arc sin}(2x - 1) + \frac{\pi}{2}$.

8. $y = 1 + \text{arc cos } 2x$.

9. $y = \frac{1}{2}\tan^{-1}(x + 1)$.

10. $y + 1 = \text{arc sin}(x - 1)$.

11. $x = \frac{\pi}{2} - \cos^{-1}(2y + 1)$.

12. $y = \frac{\pi}{2} + \frac{1}{2}\text{arc sin}(2x - 1)$.

59. Approximate solutions of equations involving trigonometric functions of one angle. The method shown in the example below is similar to the approximate solution of algebraic equations of the third or higher degrees.

Example 1. Find, correct to two decimal places, the value of x that satisfies $\sin\frac{x}{2} = 1 - \frac{x}{2}$.

From the graphs of $y = \sin\frac{x}{2}$ and $y = 1 - \frac{x}{2}$ an approximate value for x can be found, it being that value of x that makes the two values of y equal.

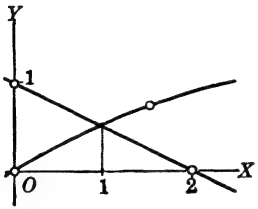


Fig. 73a

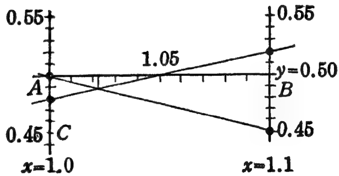


Fig. 73b

The value for x is seen from Fig. 73a to be near 1. To get a better approximation, a new table of values is made, taking x at intervals of 0.1 on each side of the approximate value of x.

x	0 9	1 0	1 1
$\sin\frac{x}{2}$	0.43	0.48	0 52
$1 - \frac{x}{2}$	0.55	0.50	0.45

This table shows that the value of x lies between 1.0 and 1.1. Plotting these values of $\sin\frac{x}{2}$ and $1 - \frac{x}{2}$ at $x = 1.0$ and 1.1, and assuming that the curve, $y = \sin\frac{x}{2}$, may be replaced by a straight line in this small interval, gives a new approximation as shown in Fig. 73b. In this graph only that portion of the curve near the point of intersection is shown. The y-axis would be to the left of A at a distance equal to ten of the intervals of AB, and the x-axis at a distance below A equal to ten intervals of CA.

From the graph the closer approximation for x is 1.02. If it is desired to check this value a new table of values can be made taking x at intervals of 0.01 on each side of 1.02.

x	1 01	1 02	1 03
$\sin\frac{x}{2}$	0 484	0 488	0 493
$1 - \frac{x}{2}$	0 495	0 490	0 485

From this table it can be seen that x lies between 1.01 and 1.02 but is nearer 1.02. If the graph of Fig. 73b had given a result nearly midway between two successive hundredths this check would be necessary in choosing between them, otherwise the result is fairly reliable.

EXERCISES

Find, correct to two decimal places, the values of x in the following:

1. $\sin x = 1 - x$.
2. $\cos \frac{x}{2} = 3x$.
3. $x + 2 + \cos x = 0$.
4. $2 \tan 3x = 1 - x$.
5. $2 \sin x = \cos x - \frac{1}{2}$.
6. $2 \sin \frac{x}{2} = \cos 4x$.
7. $\cos x = \frac{x}{2} - 1$.
8. $\cos x = \frac{x}{2} + 1$.
9. $\sin x = x^2$.
10. $\cos 2x = 2x$.

GENERAL EXERCISES

Graph the following curves between the indicated limits of x.

1. $y = 2 \sin \frac{\pi x}{2}$, -3 to 3.
2. $y = 3 \cos \frac{2x}{3}$, $-\pi$ to 2π.
3. $y = \frac{1}{2} \tan \frac{3x}{2}$, -3 to 4.
4. $y = \frac{3}{2} \sin (2x + 3)$, -4 to 2.
5. $y = \frac{5}{8} \cos \left(\frac{3x}{2} - \frac{\pi}{2}\right)$, 2 to 2π.
6. $y = 2 \sin x + \frac{1}{3} \sin 3x$, 0 to 2π.
7. $y = 2 \cos 3x - 3 \sin \pi x$, 0 to 4.
8. $y = 2 \sin \frac{x}{2} - \cos \frac{2\pi x}{5}$, $-\pi$ to $3\frac{3}{4}$.
9. $y = 2 \cos \pi x - 3 \sin 2x$, -3 to 4.
10. $y = 2 \sin x + \frac{1}{2} \sin 4x$, 0 to 2π.
11. $y = 2 \sin (x + 1.01) + \frac{1}{2} \sin 3x$, 0 to 2π.
12. $y = 2 \sin (x - 0.68) + \frac{1}{2} \sin 4x$, 0 to 2π.

Graph the following curves showing at least one period:

13. $2y = \text{arc} \cos 3x$.
14. $3y = \text{arc} \sin 2x$.
15. $3y = 2 \text{ arc} \sin \frac{x}{2}$.
16. $2x = 3 \text{ arc} \cos 2y$.
17. $y = \frac{1}{2} \text{ arc} \tan 3x$.
18. $2y = 3 \text{ arc} \sin (2x - 3) - 3$.
19. $x = 1 + \frac{2}{3} \text{ arc} \sin \frac{y}{2}$.
20. $y = 2 + \frac{1}{2} \tan^{-1} \left(\pi x + \frac{\pi}{2}\right)$.

Solve graphically, correct to 2 decimals:

21. $2x - \sin 2x = 2.$

22. $x - \frac{1}{2} = \cos x.$

23. $\sin 2x = 1 + \frac{x}{2}.$

24. $2 - 2x = \cos 2x.$

25. $\cos x = x - 2.$

26. $4 \tan \frac{x}{2} = 3 - x.$

27. Plot $y = \frac{1}{2} \sin x + \frac{\sqrt{3}}{2} \cos x$ by constructing the two curves separately and adding ordinates; also plot $y = \sin\left(x + \frac{\pi}{3}\right)$ and compare the two curves. Determine if they should be exactly the same curve.

28. Plot $y = \frac{1}{2} \cos x - \frac{\sqrt{3}}{2} \sin x$ by subtracting the ordinates of the two curves $y = \frac{1}{2} \cos x$ and $y = \frac{\sqrt{3}}{2} \sin x$; also plot $y = \cos\left(x + \frac{\pi}{3}\right)$ and compare this curve with the curve above. Determine if the two curves should be exactly the same.

29. Find the angle such that its cosine is $\frac{1}{2}$ of the radian measure of the angle. (Results to two decimal places.)

30. Find the angle such that its cosine is 2 less than its own radian measure. How many angles are there that meet the conditions?

31. The area of a segment of a circle is 10 sq. in. and its radius is 4 in. Find, in radians and correct to two decimals, the central angle subtended by the chord of the segment.

32. Find in radians, correct to two decimals, the central angle in a circle whose subtended segment is one fourth of the area of the circle.

33. In a circle whose radius is 6 in., a certain chord intercepts a segment whose area is one third of the circle. Find, correct to two decimals, the distance of the chord from the center of the circle.

34. A horizontal cylindrical tank, 6 ft. long and 3 ft. in diameter, has 6 cu. ft. of water in it. Find, correct to nearest hundredth of a foot, the depth of the water.

35. If 150 gallons of oil are poured into an empty horizontal cylindrical tank 10 ft. long and 4 ft. in diameter, find the depth of

the oil correct to the nearest hundredth of a foot. (Assume $7\frac{1}{2}$ gallons to the cubic foot.)

36. If 1000 gallons of oil are poured into an empty horizontal cylindrical tank 10 ft. long and 4 ft. in diameter, find the depth of the oil correct to the nearest hundredth of a foot. (Assume $7\frac{1}{2}$ gallons to the cubic foot.)

37. The equation $x \tan x = k$ occurs in finding the proper tones of the vibration of a loaded string. Find a value of x between 0 and $\frac{\pi}{2}$, correct to two decimal places, to satisfy this equation when (a) $k = 2$; (b) $k = 7$.

38. The calculation of the strength of a long column fixed at one end and held by a horizontal force at the other, calls for the solution of $x = \tan x$. Find the values of x between 0 and 2π, correct to two decimal places, that satisfy the equation.

39. Two pulleys, 10 ft. and 4 ft. in diameter respectively and revolving in the same direction are connected by a 40-ft. belt. Find the distance between the centers of the pulleys correct to the nearest hundredth of a foot. (It is suggested that some one angle be chosen as the variable in solving this problem.)

40. AB and BC are each tangent to a circle 5 in. in radius at A and C respectively. If the smaller arc AC plus $AB = 20$ in., find the length of AB correct to the nearest hundredth of an inch.

41. In a certain circle the tangents AB and AC are each 100 in. and the shorter arc BC is 100 in. Find, correct to the nearest tenth of an inch, the radius of the circle and the distance from A to its center.

CHAPTER VII

LOGARITHMS

60. Introduction. The long and laborious computations which frequently occur in the solution of various mathematical and practical problems can often be greatly simplified by the use of the method of calculation by logarithms. This labor-saving device reduces such fundamental operations as multiplication, division, raising to a power, and extracting a root to the more simple operations of addition, subtraction, multiplication, and division respectively.

This chapter is devoted to the theory and use of logarithms.

61. Definition of the logarithm of a number. If a number N is expressed as a power of a, so that

$$N = a^x, \tag{1}$$

then the **exponent,** x, is called the **logarithm** of N to the **base** a. Stated in words:

The logarithm of a number to a given base is the exponent by which the base must be affected to produce the number.

Symbolically, this relation is denoted by writing

$$\log_a N = x, \tag{2}$$

and is read " the logarithm of N to the base a is equal to x."

It should be noted that a logarithm is merely an *exponent.* Equations (1) and (2) state the same fact in two different ways; the former in the **exponential form** and the latter in the **logarithmic form.** This is further illustrated by the following table:

Exponential form	Logarithmic form
$3^3 = 27$	$\log_3 27 = 3$
$4^{\frac{3}{2}} = 8$	$\log_4 8 = \frac{3}{2}$
$16^{-\frac{5}{4}} = \frac{1}{32}$	$\log_{16} (\frac{1}{32}) = -\frac{5}{4}$
$13^0 = 1$	$\log_{13} 1 = 0$

EXERCISES

1. Express in logarithmic form:

(a) $5^2 = 25$.
(b) $16^{\frac{1}{2}} = 4$.
(c) $9^{\frac{3}{2}} = 27$.
(d) $4^{-0.5} = 0.5$.
(e) $(\frac{1}{8})^{\frac{2}{3}} = 0.25$.
(f) $(\frac{1}{7})^{-2} = 49$.
(g) $27^{\frac{4}{3}} = 81$.
(h) $6^{-3} = \frac{1}{216}$.
(i) $22^0 = 1$.

2. Express in exponential form:

(a) $\log_6 216 = 3$.
(b) $\log_4 (\frac{1}{2}) = -\frac{1}{2}$.
(c) $\log_{15} 1 = 0$.
(d) $\log_{16} 4 = 0.5$.
(e) $\log_{\frac{1}{8}} (\frac{1}{4}) = \frac{2}{3}$.
(f) $\log_5 0.04 = -2$.
(g) $\log_9 27 = 1.5$.
(h) $\log_{\frac{1}{7}} 49 = -2$.
(i) $\log_{27} 81 = \frac{4}{3}$.

3. Using 9 as the base, find the logarithms of the following numbers:

$$729,\ \tfrac{1}{3},\ 27,\ 1,\ \tfrac{1}{243},\ 9.$$

4. Using 8 as the base, find the logarithms of the following numbers:

$$4,\ \tfrac{1}{2},\ 1,\ \tfrac{1}{16},\ 128,\ \tfrac{1}{64}.$$

5. Find the value of each of the following logarithms:

(a) $\log_{49} 7$.
(b) $\log_4 256$.
(c) $\log_{125} 25$.
(d) $\log_{64} 0.5$.
(e) $\log_{\frac{2}{3}} (\frac{8}{27})$.
(f) $\log_\pi 1$.
(g) $\log_{27} (\frac{1}{9})$.
(h) $\log_2 2048$.
(i) $\log_{1.5} (\frac{4}{9})$.

6. Find x in each of the following equations:

(a) $\log_{\sqrt{2}} x = 4$.
(b) $\log_4 x = 1.5$.
(c) $\log_x 9 = -0.5$.
(d) $\log_x 36 = \frac{2}{3}$.
(e) $\log_{16} x = \frac{5}{4}$.
(f) $\log_{25} x = -2.5$.
(g) $\log_x 0.16 = -2$.
(h) $\log_x 2 = 0.125$.
(i) $\log_7 x = -\frac{1}{2}$.

62. Fundamental properties of logarithms. Since logarithms are merely exponents, the properties of logarithms will depend on the properties of exponents. The following index laws, which are used in the proofs of certain fundamental properties of logarithms, are restated below:

(1) $a^m \cdot a^n = a^{m+n}$,
(2) $a^m \div a^n = a^{m-n}$,
(3) $(a^m)^n = a^{mn}$,
(4) $\sqrt[n]{a^m} = a^{\frac{m}{n}}$.

The corresponding laws of logarithms may be stated as follows:

I. *The logarithm of a product is equal to the sum of the logarithms of its factors.*

Let

$$M = a^x \quad \text{and} \quad N = a^y.$$

Then from the definition of a logarithm,

$$x = \log_a M \quad \text{and} \quad y = \log_a N.$$

Multiplying,

$$MN = a^x \cdot a^y = a^{x+y}.$$

Hence,

$$\log_a MN = x + y,$$

or

$$\mathbf{\log_a MN = \log_a M + \log_a N.}$$

This law may be extended to any finite number of factors. As an illustration,

$$\log_{10} 455 = \log_{10} 5 + \log_{10} 7 + \log_{10} 13.$$

II. *The logarithm of a quotient is equal to the logarithm of the dividend minus the logarithm of the divisor.*

As above, let

$$M = a^x \quad \text{and} \quad N = a^y.$$

Then

$$x = \log_a M \quad \text{and} \quad y = \log_a N.$$

Dividing,

$$\frac{M}{N} = \frac{a^x}{a^y} = a^{x-y}.$$

Hence,

$$\log_a \frac{M}{N} = x - y,$$

or

$$\mathbf{\log_a \frac{M}{N} = \log_a M - \log_a N.}$$

As an illustration,

$$\log_7 \tfrac{301}{147} = \log_7 301 - \log_7 147.$$

III. *The logarithm of a power of a number is equal to the exponent times the logarithm of the number.*

Let

$$M = a^x, \quad \text{then} \quad x = \log_a M.$$

Raising both members to the pth power,

$$M^p = a^{px}.$$

Hence,

$$\log_a M^p = px,$$

or

$$\mathbf{\log_a M^p = p \log_a M.}$$

As an illustration,

$$\log_{13} (1.75)^5 = 5 \log_{13} 1.75.$$

IV. *The logarithm of a root of a number is equal to the logarithm of the number divided by the index of the root.*

Let

$$N = a^y, \quad \text{then} \quad y = \log_a N.$$

Extracting the qth root of both members,

$$\sqrt[q]{N} = a^{\frac{y}{q}}.$$

Hence,

$$\log_a \sqrt[q]{N} = \frac{y}{q},$$

or

$$\mathbf{\log_a \sqrt[q]{N} = \frac{1}{q} \log_a N.}$$

As an illustration,

$$\log_e \sqrt[7]{133} = \tfrac{1}{7} \log_e 133.$$

Example 1. Express $\log_8 \dfrac{\sqrt{15}\cdot 35^2}{(20)^{\frac{2}{5}}}$ in expanded form.

$$\log_8 \frac{\sqrt{15}\cdot 35^2}{(20)^{\frac{2}{5}}} = \log_8 \sqrt{15} + \log_8 35^2 - \log_8 (20)^{\frac{2}{5}}$$

$$= \tfrac{1}{2}\log_8 15 + 2\log_8 35 - \tfrac{2}{5}\log_8 20.$$

Example 2. Express $5 \log_{10} 12 - \frac{2}{7} \log_{10} 109 + \frac{1}{3} \log_{10} 38$ as a single logarithm.

$$5\log_{10} 12 - \tfrac{2}{7}\log_{10} 109 + \tfrac{1}{3}\log_{10} 38 = \log_{10} 12^5 - \log_{10}(109)^{\frac{2}{7}} + \log_{10}\sqrt[3]{38}$$

$$= \log_{10} \frac{12^5\cdot\sqrt[3]{38}}{(109)^{\frac{2}{7}}}.$$

Example 3. Evaluate: $\dfrac{\log_6 36 + \log_9 (27)^{\frac{2}{5}}}{\log_{64} (\frac{1}{8})^{1.5} - \log_{125} (25)^{0.375}}$.

$$\frac{\log_6 36 + \log_9 (27)^{\frac{2}{5}}}{\log_{64} (\frac{1}{8})^{1.5} - \log_{125} (25)^{0.375}} = \frac{\log_6 36 + \frac{2}{5}\log_9 27}{\frac{3}{2}\log_{64}(\frac{1}{8}) - \frac{3}{8}\log_{125} 25}$$

$$= \frac{(2) + (\frac{2}{5})(\frac{3}{2})}{(\frac{3}{2})(-\frac{1}{2}) - (\frac{3}{8})(\frac{2}{3})}$$

$$= \frac{2 + \frac{3}{5}}{-\frac{3}{4} - \frac{1}{4}} = \frac{\frac{13}{5}}{-1} = -\tfrac{13}{5}.$$

EXERCISES

Prove each of the following by the method used for Properties I, II, III, and IV:

1. $\log_a (P\cdot Q\cdot R) = \log_a P + \log_a Q + \log_a R.$

2. $\log_b \dfrac{P\cdot Q}{R} = \log_b P + \log_b Q - \log_b R.$

3. $\log_c \dfrac{P^n}{\sqrt[m]{R}} = n\log_c P - \dfrac{1}{m}\log_c R.$

4. $\log_e \dfrac{R\cdot\sqrt[n]{Q}}{P^m} = \log_e R + \dfrac{1}{n}\log_e Q - m\log_e P.$

Express each of the following logarithms in expanded form:

5. $\log_{10} \dfrac{\sqrt{13}}{7^5\cdot 84}$.

6. $\log_5 \dfrac{72\cdot 6^{-\frac{2}{3}}}{\sqrt[4]{18^3}}$.

7. $\log_{12} (\frac{1}{3} \pi r^2 h)$.

8. $\log_8 \dfrac{\sqrt{10} \cdot \sqrt[3]{43} \cdot \sqrt[5]{12^{-3}}}{81^3}$.

9. $\log_e \dfrac{28^2}{(100)^{-\frac{2}{3}} \cdot \sqrt[5]{219}}$.

10. $\log_{10} (\frac{4}{3} \pi r^3)$.

Express each of the following as a single logarithm:

11. $\log_{10} \pi + 3 \log_{10} d - \log_{10} 6$.

12. $2 \log_4 9 - \frac{1}{3} \log_4 17 + \frac{3}{4} \log_4 12 - \log_4 171$.

13. $\frac{1}{2} \log_a x - \frac{2}{3} \log_a y + 1.2 \log_a z$.

14. $7 \log_{10} 0.125 - \frac{2}{9} \log_{10} 1.82 - \frac{3}{5} \log_{10} 22.7 - \log_{10} 63$.

Given $\log_{10} 2 = 0.30103$, $\log_{10} 3 = 0.47712$, and $\log_{10} 7 = 0.84510$, find the logarithms of the following numbers to the base 10:

15. 42. **17.** $\frac{1}{24}$. **19.** 1.125. **21.** $\sqrt[5]{30}$. **23.** $\dfrac{48^3}{\sqrt[4]{35}}$.

16. $\sqrt[3]{189}$. **18.** $\frac{243}{32}$. **20.** $\dfrac{7}{3.375}$. **22.** 1.25^7. **24.** $\sqrt[7]{84^3}$.

In each of the following equations, express y in terms of x:

25. $\log_{10} y = x^2$.

26. $\log_e y = -x$.

27. $\log_a y = -\dfrac{1}{x}$.

28. $\log_e y = \sqrt{x} - 2 \log_e x$.

29. $\log_a y = -x^3 + \frac{1}{2} \log_a (x + 1)$.

30. $\log_{10} y = -\dfrac{1}{x^2} - \frac{3}{4} \log_{10} (1 - x^2)$.

Evaluate each of the following expressions:

31. $\dfrac{\log_{49} 7 - \log_{25} 125}{\log_e 1 + \log_{16} (\frac{1}{4})}$.

32. $\dfrac{\log_{\frac{2}{3}} (\frac{8}{27}) + \log_{27} 9}{\log_{1.5} (\frac{9}{4}) - \log_8 (0.5)}$.

33. $\dfrac{\log_5 125 - \log_8 (4)^{\frac{3}{2}}}{\log_9 \sqrt{27} + \log_3 1}$.

34. $\dfrac{\log_6 36^{-\frac{3}{2}} + \log 9^{\frac{1}{2}}}{\log_{\sqrt{7}} 1 + \log_4 8^{-0.5}}$.

35. $\dfrac{\log_3 9\sqrt{3} - \log_{64} \sqrt[5]{\frac{1}{8}}}{\frac{1}{3} \log_{125} (25^4 \cdot 5^7)}$.

36. $\dfrac{\log_3 81^{-0.15}}{\log_{49} (7^{\frac{3}{2}} \div 49\sqrt{7})}$.

37. $\dfrac{\log_{64} (\frac{1}{32})^{0.125} \cdot \log_6 1}{\log_{81} 3\sqrt{3}}$.

38. $\dfrac{\log_{2\sqrt{2}} 8^{-\frac{2}{3}} + 2 \log_{27} (\frac{1}{9})^{0.75}}{\log_{10} (0.0001) - \log_{0.36} \sqrt{\frac{3}{5}}}$.

39. If a sequence of numbers are in geometrical progression, show that their corresponding logarithms are in arithmetical progression.

63. Systems of logarithms. While any positive number except unity may be used as a base, there are only two bases in common use.

The **Natural** or **Naperian System of Logarithms,** introduced by John Napier (1614), employs the irrational number $e(= 2.7182818 \cdots$ to seven decimals) for its base. This system is of extreme importance for theoretical purposes in higher mathematics and will be met by the student in the study of the calculus.

The **Common** or **Briggsian System of Logarithms,** named after its inventor Henry Briggs (1616), employs the base 10. This system is more convenient for computational purposes and is the one commonly used.

In this book, when the base is not expressed, the base 10 is understood. Thus $\log N$ is understood to mean $\log_{10} N$. As an abbreviation for $\log_e N$, $\ln N$ is often employed.

EXERCISES

1. Why cannot 1 be used as a base for a system of logarithms?

2. Why is it that a negative number cannot be used as a base for a system of logarithms?

3. Why is it impossible to find the logarithm of a negative number to a positive base?

64. Characteristic and mantissa of a logarithm. Consider the following table in which 10 is taken as the base:

Exponential form	Logarithmic form
$10^4 = 10{,}000$	$\log 10{,}000 = 4$
$10^3 = 1000$	$\log 1000 = 3$
$10^2 = 100$	$\log 100 = 2$
$10^1 = 10$	$\log 10 = 1$
$10^0 = 1$	$\log 1 = 0$
$10^{-1} = 0.1$	$\log 0.1 = -1$ or $9 - 10$
$10^{-2} = 0.01$	$\log 0.01 = -2$ or $8 - 10$
$10^{-3} = 0.001$	$\log 0.001 = -3$ or $7 - 10$
$10^{-4} = 0.0001$	$\log 0.0001 = -4$ or $6 - 10$

It is evident from the above table that the logarithm of a positive or negative integral power of 10 is respectively a positive or negative number. The logarithms of all other positive numbers consist of an integral and a decimal part. The integral part of a logarithm is called its **characteristic,** and the decimal part is called its **mantissa.**

For example, the logarithm of any number between 1000 and 10,000, that is any number which consists of 4 digits to the left of the decimal point, must lie between 3 and 4, and may be written 3 + a decimal. Similarly, considering only the number of digits to the left of the decimal point, the logarithm of a 3 digit number is 2 + a decimal, of a 2 digit number 1 + a decimal, and of a 1 digit number 0 + a decimal. Hence the rule:

If a number is greater than 1, *the characteristic of its logarithm is positive,* and is one less than the number of digits to the left of the decimal point.*

Now consider the logarithms of numbers less than 1. If a number lies between 0.1 and 1, its logarithm lies between -1 and 0, which may be written as $-1 +$ a decimal or $0 -$ a decimal. For convenience in computing, it is desirable to select the decimal part as positive, hence -1 or its equivalent $9 - 10$ is taken as the characteristic. Similarly, the logarithm of a number between 0.01 and 0.1 is $-2 +$ a decimal or $8 +$ a decimal $- 10$. Continuing in this way, it is clear that the characteristic of a number having two zeros immediately following the decimal point is -3 or $7 - 10$, and so on. Hence the rule:

If a number is less than 1, *the characteristic of its logarithm is negative, and is* 9 *minus the number of zeros immediately following the decimal point minus* 10.

Since most logarithms are non-repeating infinite decimal fractions, the mantissa or decimal part can in general be only approximated. This can be obtained directly from

* Zero is considered here as a positive number.

tables of mantissas, called **Tables of Logarithms**, which have been calculated to various degrees of accuracy and are known as four-place tables, five-place tables, etc. according to the number of digits in the mantissa.

Since,

$$\begin{aligned}
\log(N \cdot 10) &= \log N + \log 10 = \log N + 1,\\
\log(N \cdot 10^2) &= \log N + \log 10^2 = \log N + 2,\\
&\cdots\cdots\cdots\cdots\cdots,\\
\log(N \cdot 10^n) &= \log N + \log 10^n = \log N + n,
\end{aligned}$$

and

$$\begin{aligned}
\log(N \div 10) &= \log N - \log 10 = \log N - 1,\\
\log(N \div 10^2) &= \log N - \log 10^2 = \log N - 2,\\
&\cdots\cdots\cdots\cdots\cdots,\\
\log(N \div 10^n) &= \log N - \log 10^n = \log N - n,
\end{aligned}$$

where n is any positive integer, it is evident that if a number is multiplied or divided by an integral power of 10, an integer is added to, or subtracted from its logarithm. Hence the theorem:

The logarithms of numbers which differ only in the position of the decimal point have the same mantissa.

EXERCISE

1. Write down the characteristics of the logarithms of the following numbers: (a) 49.167; (b) 0.16823; (c) 2.0698; (d) 0.0031894; (e) 0.00094005; (f) 86,100; (g) 387.99; (h) 0.074318; (i) 40.002; (j) e; (k) 4π.

65. Use of tables. Tables of logarithms are used to get the mantissas of the logarithms of numbers and to find the numbers corresponding to given logarithms. Four-place and five-place tables have the widest use, but tables have been computed correct to six, seven, and even more decimal places, the methods of computing being beyond the scope of this book. The principles involved in using the different tables are the same. In the discussions and problems following, the use of a five-place table is assumed.

To get the logarithm of a number, its characteristic and its mantissa must be found and the two operations are distinct. The characteristic depends only on the location of the decimal point and can be determined by the rules of Art. 64. In using tables to find mantissas, the process differs according to whether the mantissa appears directly in the tables or must be found by an approximation process called **interpolation.** In general, five-place tables give mantissas directly for numbers of four digits, and four-place tables the mantissas of numbers of three digits, but tables differ widely. While mantissas are decimals, they are given in the tables without the decimal point. In the discussions following, they are often written without a decimal point, but care must be taken to include any part of the mantissa that has meaning when it is written with the decimal point. As the mantissa of a number has been shown to be independent of the position of the decimal point, the decimal point in a number is often omitted in looking up mantissas, and the result called **a sequence of digits.**

The portion of the five-place table given below is used in the illustrative problems which follow.

300 — Logarithms of Numbers — 350

N.	0	1	2	3	4	5	6	7	8	9
300	47 712	727	741	756	770	784	799	813	828	842
01	857	871	885	900	914	929	943	958	972	986
02	48 001	015	029	044	058	073	087	101	116	130
03	144	159	173	187	202	216	230	244	259	273
04	287	302	316	330	344	359	373	387	401	416
05	430	444	458	473	487	501	515	530	544	558
06	572	586	601	615	629	643	657	671	686	700
07	714	728	742	756	770	785	799	813	827	841
08	855	869	883	897	911	926	940	954	968	982
09	996	*010	*024	*038	*052	*066	*080	*094	*108	*122
310	49 136	150	164	178	192	206	220	234	248	262

Prop. Pts.	15	14
1	1.5	1.4
2	3.0	2.8
3	4.5	4.2
4	6.0	5.6
5	7.5	7.0
6	9.0	8.4
7	10.5	9.8
8	12.0	11.2
9	13.5	12.6

The column headed N gives the first three digits, reading from the left, of the number and the fourth is at the top of the table. The first two digits of the row of mantissas are to be repeated in connection with each mantissa until another complete mantissa is given. An asterisk indicates that the first two digits of the mantissa are to be found in the row following instead of in the preceding rows.

The process of finding the logarithm of a number of four digits will be shown in Example 1 following, and of a number of five digits in Example 2. In the latter problem interpolation must be used. After the process has been shown, the reasoning on which it is based will be discussed.

Example 1. Find log 0.003035.

By the rules of Art. 64, the characteristic is 7 − 10. To find the mantissa, first glance down the column headed N to find the first three digits, reading from the left, then at the top of the table for the fourth digit. In the row with 303 and column headed 5 is found 48216.

Hence $\log 0.003035 = 7.48216 - 10.$

Example 2. Find log 30.644.

By the rules of Art. 64, the characteristic is 1. As the number contains five digits, its mantissa is not recorded in the table. The mantissas of the next lower and higher numbers, 30640 and 30650, are found as in Example 1, to be 48629 and 48643 respectively. The difference between these two adjacent mantissas, called the **tabular difference,** is 14. Since 30644 is four-tenths of the interval from 30640 to 30650, four-tenths of 14 is added to the mantissa of 30640. This part of the tabular difference is called the **correction** and the nearest integer only is used. The statements above may be summarized as follows:

N	mantissa				
30650	48643	mantissa	30640	=	48629
30640	48629	correction	(0.4 × 14)		6
tabular difference	= 14	mantissa	30.644	=	48635
		log 30.644		=	1.48635

The process shown in Example 2 of finding the mantissa of a number where the mantissa lies between two values in the

table, is called **interpolating.** In this process it is assumed that for small differences in numbers, the change in the mantissa is proportional to the change in the number. This is not always true, but the results are nearly always correct to the same number of decimal places as are given in the table used.

In any particular problem where the logarithm of a number is required, only the complete logarithm (which includes both characteristic and mantissa, interpolated if necessary) should be shown. The student will find logarithms of little service until the complete logarithm can be obtained directly from the table, with the intermediate steps performed mentally. Tables of proportional parts are an aid in interpolating and are included in some tables. Thus, in the portion of the table shown above, the numbers 15 and 14 under the title "Prop. Pts." are the tabular differences corresponding to that portion of the table. The numbers 1, 2, 3, $\cdots$ 9, in the vertical column to the left, are the tenths, and under 15 and 14 are their products by the tenths. This table gives then all possible corrections for these tabular differences. Thus in Example 2, four-tenths of 14 could have been found by taking the number opposite 4 in the column headed 14. Another problem is added to further illustrate the process of finding the logarithm of a number.

Example 3. Find log 309.46.

The characteristic is 2; the mantissa of 30940 = 49052; the tabular difference = 14; the correction $(0.6 \times 14) = 8$. Hence the mantissa is $49052 + 8$ or 49060.

$$\therefore \log 309.46 = 2.49060.$$

EXERCISES

Verify the following using a five-place table:

1. $\log 76.932 = 1.88611.$
2. $\log 629.54 = 2.79902.$
3. $\log 9.8351 = 0.99278.$*
4. $\log 7.5516 = 0.87804.$
5. $\log 0.010072 = 8.00312 - 10.$
6. $\log 0.045001 = 8.65322 - 10.$

* This logarithm might also be 0.99277. In this book whenever the correction is equally near two integers, the larger is used.

7. $\log 89987 = 4.95418$.
8. $\log 0.0033183 = 7.52092 - 10$.
9. $\log 1.0003 = 0.00013$.
10. $\log 222.32 = 2.34698$.

Find the following using a five-place table:

11. $\log 0.75879$.
12. $\log 8.0008$.
13. $\log 989.48$.
14. $\log 0.62541$.
15. $\log 0.42719$.
16. $\log 2.6306$.
17. $\log 29.099$.
18. $\log 0.00030009$.
19. $\log 0.016437$.

The process of finding from tables the number corresponding to a given logarithm is the inverse of the process described above and the number is called the **anti-logarithm.** Since the characteristic of a logarithm depends only on the location of the decimal point, and the mantissa only on the sequence of the digits, then in a given logarithm the sequence of digits will be determined from the given mantissa and the decimal point from the given characteristic. The process will be shown by examples.

Example 4. Find x if $\log x = 2.49206$.
First look in the table for 49206 or for two adjacent mantissas between which it lies. In this case 49206 is found in the table, the corresponding number in the N column is 310 and the number at the top of the table is 5. Hence the first four digits are 3105 and the fifth is 0, no interpolation being required to find the fifth digit when the mantissa appears directly in the table. The characteristic 2 indicates that the number has three digits to the left of the decimal point.
Therefore if $\log x = 2.49206$, $x = 310.50$.*

Example 5. Find x if $\log x = 1.48307$.
This mantissa is not recorded in the tables but lies between two adjacent mantissas, 48302 and 48316, of the tables, the corresponding numbers being 30410 and 30420, respectively. By the same assumption as was made in interpolation, x, without regard to decimal point, lies between 30410 and 30420 and divides the interval between them in the same ratio as 48307 divides the interval from

* The difference in meaning between 310.50 and 310.5 will be discussed in Art. 66.

48302 to 48316. The tabular difference is 14, and the difference between the given mantissa 48307 and the next lower one in the table is 5. Hence x lies $\frac{5}{14}$ of the interval from 30410 to 30420. The integer nearest to $\frac{5}{14}$ of 10 is 4 and is the fifth digit. Hence the sequence of digits is 30414. This fifth digit can be more easily obtained from the table of proportional parts. Under the column headed 14 find the number nearest 5, in this case 5.6. The number in the corresponding left-hand column is 4 and is the fifth digit. The statements above may be summarized:

Mantissa	N		
48316	30420	given mantissa	= 48307
48302	30410	next lower mantissa	= 48302
14 = tabular difference		correction	= 5
		fifth digit ($\frac{5}{14} \times 10$) =	4

A characteristic of 1 indicates two digits to the left of the decimal point.

Hence if $\log x = 1.48307$, $x = 30.414$.

The student should be able to perform mentally all the necessary work in getting a number from its logarithm. The necessary work, without so detailed an explanation, is given in another example below.

Example 6. Find x if $\log x = 7.47739 - 10$.

The first four digits are 3001; the tabular difference is 14, the correction is 12. Under 14 in the table of proportional parts the number nearest 12 is 12.6 and the tenth opposite is 9, hence the digits are 30019. A characteristic $7 - 10$ indicates two zeros between the decimal point and the first digit.

Hence $x = 0.0030019$.

EXERCISES

Verify the following from tables:

1. If $\log x = 0.65948$, $x = 4.5654$.
2. If $\log x = 9.69676 - 10$, $x = 0.49746$.
3. If $\log x = 1.45932$, $x = 28.795$.
4. If $\log x = 8.89008 - 10$, $x = 0.077638$.

5. If $\log x = 2.21078$, $x = 162.47$.
6. If $\log x = 7.26844 - 10$, $x = 0.0018554$,
7. If $\log x = 0.05474$, $x = 1.1343$.
8. If $\log x = 3.00033$, $x = 1000.8$.

Find x in each of the following:

9. $\log x = 0.68445$.
10. $\log x = 9.63169 - 10$.
11. $\log x = 2.77378$.
12. $\log x = 7.99103 - 10$.
13. $\log x = 1.86008$.
14. $\log x = 5.52058 - 10$.
15. $\log x = 5.36284$.
16. $\log x = 8.39080 - 10$.
17. $\log x = 4.00015$.

66. Approximations and significant figures. As the logarithms are, in most cases, approximations, any results obtained from them will likewise be only approximately true, and it is often of importance to know what accuracy can be expected from calculations made under such circumstances. In addition, the numbers to which the logarithms are applied may in themselves be only approximations. For instance if the data results from the measurement of a line, the length is only accurate within certain limits due to the limitations of the instruments used in measuring. The accuracy of a number is usually indicated by stating its number of **significant figures.** In this zeros used only to put other digits in their proper position as regards the decimal point are not considered significant. However one or more zeros coming after the decimal point and at the end of a sequence of other digits are considered significant. Thus 310.50 has five significant figures, while 310.5 has four. Considered as an approximation 310.50 means any number nearer to 310.50 than to 310.49 or to 310.51, 310.5 any number nearer to 310.5 than to 310.4 or to 310.6. When a surveyor calls a length 110.10 feet he means that it lies between 110.09 feet and 110.11 feet and is nearer to 110.10 than to either 110.09 or 110.11. The number of significant figures of several given numbers is indicated in the following table.

Number	significant figures	Number	significant figures
132.04	5	0.01452	4
12001	5	0.00415261	6
132.00	5	0.000014500	5
13200	3 or 5	0.145500000	9

Some confusion may arise with numbers having zeros following other digits and to the left of the decimal point. Thus 13200 may represent a number nearer 13200 than to 13300 or to 13400 in which case it has three significant figures; if it represents a number nearer to 13200 than to 13199 or 13201, it has five significant figures. The latter can be indicated by 13200, correct to five significant figures, or the context may indicate the accuracy of the number. In many problems when integers are used with other data of five-place accuracy they are assumed to be of that same degree of accuracy. In this text, when numbers of more than five digits appear the nearest number of five digits is to be used; thus, for 486.236, use 486.24; for 486.234, use 486.23; and for 78.911864 use 78.912. For convenience it has been assumed that all data in the problems of this book is accurate enough to warrant the use of a five-place table.

67. Computation by means of logarithms. The application of logarithms to shorten calculations depends on their properties as given in Art. 62, and the processes where they are of particular service have been mentioned in Art. 60. However the limitations placed on logarithms by reason of the tables used must be remembered. Thus in multiplying together by logarithms, two numbers each of five significant figures, only five significant figures will appear in the product in place of the nine or ten that may appear in direct multiplication. The accuracy of the tables to be used in a given problem depends on the accuracy of the given data. In general, the number of digits in the mantissa is the same as the number of significant figures in the given data.

The examples below illustrate some possible computations

by logarithms. Attention is called to the advantage, or even necessity, of a careful arrangement of the work. Some outline showing where each logarithm will be written and each computation made, should be made before looking up any of the logarithms. The following arrangement which can be modified to meet the needs of each problem is suggested to the student. The first column indicates the operation, the second gives the original logarithm, the third the logarithm resulting from the original operation, and the fourth any required anti-logarithm. This arrangement will be illustrated by the following examples. While in these illustrative examples some notes may be added to call attention to certain principles, these are not a part of the solution.

Example 1. Evaluate to five significant figures:

$$\frac{(30.472)\ (0.068741)}{0.99488}.$$

If x equals the given fraction, then by taking the logarithm of both sides of the equation and employing properties I and II of Art. 62,

$$\log x = \log 30.472 + \log 0.068741 - \log 0.99488.$$

Hence it is only necessary to get the logarithms of the various numbers and combine as indicated. The anti-logarithm of the result is the required number. The problem and its solution appear below:

Let $$x = \frac{30.472 \times 0.068741}{0.99488},$$

then $$\log x = \log 30.472 + \log 0.068741 - \log 0.99488.$$

Indicated operation	original log	derived log	anti-log
log 30.472	1.48390		
log 0.068741	8 83722−10		
log numerator		0.32112	
log 0.99488	9 99777−10		
log x		0.32335	
x			2.1055

Logarithms may be used to raise numbers to required powers or to extract required roots. As the definition of a logarithm has been limited to the logarithms of positive numbers, the sign of the result must be obtained independent of the logarithmic calculation, and the numerical value then obtained by operations on positive integers. For example, if $(-0.00426)^3$ is required, it would be changed to the form $-(0.00426)^3$, the value of $(0.00426)^3$ then obtained by logarithms, and a negative sign prefixed to the result. The processes of raising to powers and of extracting roots are shown in Examples 2 and 3 below:

Example 2. Evaluate $(0.025793)^5$ to five significant figures.

Let $x = (0.025793)^5$, then $\log x = 5 \log (0.025793)$.

Indicated operation	original log	derived log	anti-log
5 log (0.025793)	8.41150−10	2 05750−10	0.000000011416

$$\begin{array}{r} 8.41150-10 \\ 5 \\ \hline \end{array}$$

$$42\ 05750 - 50 = 2.05750 - 10$$

Example 3. Evaluate $\sqrt[3]{-0.0064213}$ to five significant figures.

Let $x = \sqrt[3]{0.0064213}$, then $\log x = \frac{1}{3} \log 0.0064213$.

Indicated operation	original log	derived log	anti-log
$\frac{1}{3}$ log 0.0064213	7 80762−10	9.26921−10	0 18587

$$3\overline{)27\ 80762-30^*}$$
$$9.26921-10$$

$$\therefore \sqrt[3]{-0.0064213} = -0.18587.$$

* Before dividing any logarithm involving a negative characteristic by any number, it is well to change the logarithm to such an equivalent form that the quotient involves − 10. If the divisor had been 5 in this problem, the logarithm would have been changed to the form 47.80762 − 50.

Example 4. Evaluate to five significant figures:

$$\frac{\sqrt[5]{0.097004} + \sqrt[3]{69.321}}{(0.077624)^3}.$$

Before taking the logarithm of this fraction, it is necessary to evaluate each of the terms in the numerator and add, as the logarithm of an algebraic sum cannot be found directly. When the two terms of the numerator have been added, the logarithm of the fraction can be found. The values of each of these two terms can be found by the use of logarithms.

Indicated operation	original log	derived log	anti-log
$\frac{1}{5}$ log 0.097004	8.98679−10	9.79736−10	0.62713
$\frac{1}{3}$ log 69.321	1.84087	0.61362	4.1079
numerator			4.73503 or 4.7350
log numerator	0.67532	10.67532−10	
3 log 0.077624	8.88999−10	6.66997−10	
log fraction		4.00535	
number			10124
$5\overline{)48.98679-50}$ $9.797358-10$	$3\overline{)1.84087}$ 0.613623	$8.88999-10$ $\underline{3}$ $26.66997-30$	

Example 5. Evaluate $(0.34012)^{-\frac{1}{2}}$ to five significant figures.

Let $x = (0.34012)^{-\frac{1}{2}}$, then $\log x = -\frac{1}{2}\log 0.34012$, $\log 0.34012 = 9.53163 - 10$ and $-\frac{1}{2}\log 0.34012 = -4.76582 + 5$, where the minus sign in the logarithm affects the mantissa as well as the characteristic. As the mantissas are given as positive numbers in the tables, it is necessary to change this logarithm to an equivalent form whose mantissa is positive. This logarithm is a positive number so an equivalent form can be found by subtraction. $5 - 4.76582 = 0.23418$ and the logarithm has been changed to a form where the mantissa is a positive number.

Indicated operation	original log	derived log	anti-log
$-\frac{1}{2}$ log 0.34012	9.53163−10	0.23418	1.7147

$$-2)\underline{19.53163 - 20}$$
$$-9.76582 + 10 = 0\ 23418$$

This problem can also be solved by simplifying as follows:

$$(0.34012)^{-\frac{1}{2}} = \left[\frac{1}{0.34012}\right]^{\frac{1}{2}}.$$

Indicated operation	original log	derived log	anti-log
log 1	0 00000		
log 0.34012	9.53163−10		
difference in logs		0.46837	
$\frac{1}{2}$ difference in logs		0 23418	
number			1.7147

Example 6. Evaluate $(3401.2)^{-\frac{1}{2}}$ to five significant figures.

In this problem, the product of $-\frac{1}{2}$ and the logarithm gives a negative number, −1.76582. To get an equivalent form where the mantissa is positive, write 10 − 1.76582 − 10 = 8.23418 − 10. Here the mantissa is positive. Then the anti-logarithm can be found. The solution is shown below.

Let $\quad x = (3401.2)^{-\frac{1}{2}}, \quad$ then $\quad \log x = -\frac{1}{2} \log 3401.2.$

Indicated operation	original log	derived log	anti-log
$-\frac{1}{2}$ log 3401.2	3.53163	8.23418−10	0 017147

$$-2)\underline{3.53163} \qquad \begin{array}{r} 10.00000-10 \\ \underline{-1.76582} \\ 8\ 23418-10 \end{array}$$
$$-1.76582$$

This problem can also be solved by replacing $(3401.2)^{-\frac{1}{3}}$ by $\left[\frac{1}{3401.2}\right]^{\frac{1}{3}}$ and evaluating as shown in Example 5.

Example 7. Evaluate to five significant figures:

$$[109.09]^{0.2} + [0.062318]^{1.04}.$$

Indicated operation	original log	derived log	anti-log
0.2 log 109 09	2.03778	0.40756	2.5560
1.04 log 0.062318	8 79462−10	8.74640−10	0.055770
number			2.61177 or 2.6118

$$\begin{array}{r} 2.03778 \\ 0\ 2 \\ \hline 0.407556 \end{array} \qquad \begin{array}{r} 98.79462-100 \\ 1.04 \\ \hline 39517848-104 \\ 9879462 \\ \hline 102\ 7464048-104 \end{array} = 8.74640-10$$

Example 8. Evaluate to five significant figures:

$$\frac{(0.058627)^{-\frac{1}{2}}}{6-(5.7326)^{-0.6}}.$$

Indicated operation	original log	derived log	anti-log
$-\frac{1}{2}$ log 0 058627	8.76810−10	0.61595	
−0 6 log 5.7326	0.75835	9.54499−10	0.35074
denominator			5.6493
log numerator		10 61595−10	
log denominator		0.75199	
difference in logs		9.86396−10	
number			0.73107

$$-2)\underline{18.76810-20} \qquad \begin{array}{r} 0.75835 \\ -0\ 6 \\ \hline -0\ 455010 \end{array} = 9.54499-10$$

$$-9.38405+10 = 0.61595$$

EXERCISES

Evaluate to five significant figures:

1. $(0.063134)\ (7.2089)\ (0.51277)$.
2. $(0.43210)\ (968.43)\ (0.00042133)$.
3. $(0.22917)\ (3.0005) \div 0.025722$.
4. $(34.210)\ (6.3298) \div 421.04$.
5. $4215.7 \div (-82.761 \times 426.59)$.
6. $0.047869 \div (0.084251 \times 0.00025759)$.
7. $(2563.8)\ (-3.4419) \div (714.76 \times 0.51104)$.
8. $0.061676 \times 6.7696 \div (79.489 \times 0.052005)$.
9. $(4.4324)^4$.
10. $(3.4211)^3$.
11. $(-0.0043007)^2$.
12. $(0.89421)^5$
13. $(3.8642)^{\frac{1}{3}}$.
14. $(-7.2438)^{\frac{1}{5}}$.
15. $(-0.043007)^{\frac{1}{3}}$.
16. $(0.91288)^{\frac{1}{5}}$.
17. $(0.89154)^{0.3}$.
18. $(4.3281)^{0.4}$.
19. $\sqrt[3]{0.0078965} \div (1.3457)^7$.
20. $(56.333)^{\frac{2}{3}} \div \sqrt{11.119}$.
21. $\dfrac{12.396 \times (0.52364)^{\frac{2}{3}}}{(-52.367)^{\frac{3}{5}}}$.
22. $\dfrac{(32.145)^{[illegible]}}{(2.4563)^{\frac{1}{3}} \cdot (38.642)}$.
23. $\sqrt{\dfrac{3.1423 \times 0.52367}{(85.909)^{\frac{2}{3}}}}$.
24. $\dfrac{\sqrt[3]{8.1923} \cdot \sqrt{0.062845}}{0.98349}$.
25. $\sqrt[3]{186.21^2 - 108.26^2}$.

HINT. $186.21^2 - 108.26^2 = (186.21 + 108.26)\ (186.21 - 108.26)$.

26. $\sqrt[3]{142.71^2 - 204.46^2}$.

27. If $d = 0.02758\sqrt{DL\sqrt{p}}$; find d when $D = 30.964$, $L = 75.673$, and $p = 150.81$.

28. If $q = \dfrac{8\,c}{15} H^{\frac{3}{2}} \sqrt{2\,g}$; find q when $c = 0.59202$, $H = 0.30000$, and $g = 32.200$.

29. $\dfrac{(0.0067854)^{\frac{2}{3}} - \sqrt[3]{0.0078965}}{1.3457}$.

30. $\dfrac{(0.0076854)^{\frac{2}{3}} - \sqrt[3]{0.0087965}}{(1.3457)^7}$.

31. $950.03^{-0.2}$.
32. $3.0031^{-0.3}$.
33. $(-77.628)^{-\frac{1}{3}}$.
34. $(-9628.4)^{-\frac{1}{5}}$.

35. $0.0074260^{-\frac{1}{2}}$.

36. $0.077628^{-\frac{1}{4}}$.

37. $0.030031^{-0.3}$.

38. $0.095003^{-0.2}$.

39. $[0.095003^{-0.2} + 0.89154^{0.8}]^{0.54}$.

40. $\sqrt{16.236^{-0.6} + 0.0010078^{-\frac{1}{3}}}$.

41. $\left[\dfrac{4 - 0.056224^{-0.2}}{25.623}\right]^3$.

42. $\dfrac{0.046982^{-\frac{1}{3}}}{4 - 0.57326^6}$.

43. $\dfrac{6.8578^{-\frac{1}{4}}}{2 - 0.057231^{-0.7}}$.

44. $\dfrac{6.5878^{-\frac{1}{4}}}{2 - 0.052731^{-0.7}}$.

45. $\sqrt[4]{\dfrac{45.732^{-0.21}}{(-5.0004)^{\frac{1}{3}} + (0.0044053)^{-0.2}}}$.

46. $\dfrac{0.67654^{-0.4} + 0.94136^{0.3}}{2.0527^{-\frac{1}{3}}}$.

47. $\dfrac{0.076367^{-0.4} + 0.0016513^{0.7}}{0.87705 - 3.0004^{-\frac{1}{3}}}$.

48. $\dfrac{0.22917^{-0.31} - \sqrt[3]{3.0005}}{0.025737^{0.42}}$.

49. $\sqrt[5]{\dfrac{(0.097603)^{-0.6} + (0.12003)^{1.2}}{(1.4004)^{-\frac{1}{3}}}}$.

50. $[5 + 0.26943^{0.41}]^{-1.4}$.

51. $\dfrac{12.683^{\frac{1}{2}} + 0.0027654^{-0.3}}{18.679^2}$.

52. $\dfrac{5.8621^{\sqrt{2}} + 6.4315^{\sqrt{3}}}{8.4321^{-\sqrt{2}}}$.

53. The volume of the portion of a sphere included between two parallel planes is given by the formula $V = \dfrac{\pi h}{6}(3r^2 + h^2)$ where h is the distance between the planes and r is the radius of the sphere. Find the value of V when

(a) $r = 10.021$, $h = 6.4828$; (b) $r = 6.4828$, $h = 10.021$.

54. The number of r.p.m. of a certain type of water turbine is given by $n = \dfrac{400}{6^{1.3}} \cdot h^{1.3} \cdot P^{-0.4}$ where h is the height of the fall in feet,

and P is the horse power developed. Find n when $h = 13$ ft. and $P = 86$.

55. The time, T, of oscillation of a simple pendulum of length L is given by the formula, $T = \pi\sqrt{\dfrac{L}{g}}$. If $g = 32.161$, and $L = 3.3267$, find T.

56. The amount, S, of an annuity of 9 dollars per year payable in p equal installments, is given by $S = \dfrac{a[(1 + i)^n - 1]}{p[(1 + i)^{\frac{1}{p}} - 1]}$ where n equals the number of years and i is the yearly rate of interest. If $i = 4\%$, $n = 12$, $p = 4$ and $a = \$120$, find S.

57. The present value, A, of an annuity of a dollars a year payable in p equal installments is given by $A = a \cdot \dfrac{1 - (1 + i)^{-n}}{p[(1 + i)^{\frac{1}{p}} - 1]}$, where n = number of years and i is the yearly rate of interest. Find A if $i = 4\%$, $n = 20$, $p = 4$, and $a = \$180$.

SUMMARY OF FORMULAS

Arc of a circle expressed in terms of its radius and central angle.

[1] $s = r\theta.$

Reciprocal relations.

[2] $\csc\theta = \dfrac{1}{\sin\theta}$ and $\sin\theta = \dfrac{1}{\csc\theta}.$

[3] $\sec\theta = \dfrac{1}{\cos\theta}$ and $\cos\theta = \dfrac{1}{\sec\theta}.$

[4] $\operatorname{ctn}\theta = \dfrac{1}{\tan\theta}$ and $\tan\theta = \dfrac{1}{\operatorname{ctn}\theta}.$

Pythagorean relations.

[5] $\sin^2\theta + \cos^2\theta = 1.$

[6] $1 + \tan^2\theta = \sec^2\theta.$

[7] $1 + \operatorname{ctn}^2\theta = \csc^2\theta.$

Quotient relations.

[8] $\tan\theta = \dfrac{\sin\theta}{\cos\theta}.$

[9] $\operatorname{ctn}\theta = \dfrac{\cos\theta}{\sin\theta}.$

Area of a triangle in terms of two sides and the included angle.

[10a] $K = \frac{1}{2}bc\sin A.$

[10b] $K = \frac{1}{2}ac\sin B.$

[10c] $K = \frac{1}{2}ab\sin C.$

Addition formulas.

[11] $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$

[12] $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$

[13] $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta.$

[14] $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$

[15] $\tan(\alpha + \beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}.$

[16] $\tan(\alpha - \beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}.$

Double angle formulas.

[17] $\sin 2\alpha = 2\sin\alpha\cos\alpha.$

[18a] $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha.$

[18b] $\cos 2\alpha = 1 - 2\sin^2\alpha.$

[18c] $\cos 2\alpha = 2\cos^2\alpha - 1.$

[19] $\tan 2\alpha = \dfrac{2\tan\alpha}{1 - \tan^2\alpha}.$

Half-angle formulas.

[20] $\sin\dfrac{\alpha}{2} = +\sqrt{\dfrac{1 - \cos\alpha}{2}}$ or $-\sqrt{\dfrac{1 - \cos\alpha}{2}}.$

[21] $\cos\dfrac{\alpha}{2} = +\sqrt{\dfrac{1 + \cos\alpha}{2}}$ or $-\sqrt{\dfrac{1 + \cos\alpha}{2}}.$

[22a] $\tan\dfrac{\alpha}{2} = +\sqrt{\dfrac{1 - \cos\alpha}{1 + \cos\alpha}}$ or $-\sqrt{\dfrac{1 - \cos\alpha}{1 + \cos\alpha}}.$

[22b] $\tan\dfrac{\alpha}{2} = \dfrac{1 - \cos\alpha}{\sin\alpha}.$

[22c] $\tan\dfrac{\alpha}{2} = \dfrac{\sin\alpha}{1 + \cos\alpha}.$

Algebraic sum of sines and cosines expressed as products.

[23] $\sin P + \sin Q = 2\sin\frac{1}{2}(P + Q)\cos\frac{1}{2}(P - Q).$

[24] $\sin P - \sin Q = 2\cos\frac{1}{2}(P + Q)\sin\frac{1}{2}(P - Q).$

[25] $\cos P + \cos Q = 2\cos\frac{1}{2}(P + Q)\cos\frac{1}{2}(P - Q).$

[26] $\cos P - \cos Q = -2\sin\frac{1}{2}(P + Q)\sin\frac{1}{2}(P - Q).$

Law of sines.

[27] $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Law of cosines.

[28a] $a^2 = b^2 + c^2 - 2\,bc \cos A.$
[28b] $b^2 = a^2 + c^2 - 2\,ac \cos B.$
[28c] $c^2 = a^2 + b^2 - 2\,ab \cos C.$

Law of tangents.

[29a] $$\frac{a+b}{a-b} = \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)}.$$

[29b] $$\frac{a+c}{a-c} = \frac{\tan \frac{1}{2}(A+C)}{\tan \frac{1}{2}(A-C)}.$$

[29c] $$\frac{b+c}{b-c} = \frac{\tan \frac{1}{2}(B+C)}{\tan \frac{1}{2}(B-C)}.$$

Half-angle formulas in terms of the sides of a triangle.

$$2\,s = a + b + c \quad \text{and} \quad r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

[30a] $$\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}.$$

[30b] $$\sin \frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{ac}}.$$

[30c] $$\sin \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$$

[31a] $$\cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}.$$

[31b] $$\cos \frac{B}{2} = \sqrt{\frac{s(s-b)}{ac}}.$$

[31c] $$\cos \frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}.$$

[32a] $\tan\frac{A}{2} = \frac{r}{s-a}$.

[32b] $\tan\frac{B}{2} = \frac{r}{s-b}$.

[32c] $\tan\frac{C}{2} = \frac{r}{s-c}$.

Area of a triangle expressed in terms of its sides.

$2s = a + b + c$ and $r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$.

[33] $K = \sqrt{s(s-a)(s-b)(s-c)}$.

[34] $K = sr$.

Sector and segment areas of a circle.

[35] $K(\text{sector}) = \frac{1}{2} r^2\theta$.

[36] $K(\text{segment}) = \frac{1}{2} r^2(\theta - \sin\theta)$.

Table I — Radian Measure — Trigonometric Functions

θ Rad.	Sin θ	Cos θ	Tan θ	θ Rad.	Sin θ	Cos θ	Tan θ	θ Rad.	Sin θ	Cos θ	Tan θ
.00	.000	1 000	.000	**55**	523	853	613	**1 10**	891	.454	1 96
.01	010	1 000	010	56	.531	847	.627	1 11	896	.445	2 01
.02	020	1 000	.020	57	.540	842	.641	1 12	.900	.436	2 07
.03	030	1 000	.030	58	.548	.836	.655	1.13	904	.427	2 12
.04	040	.999	.040	59	556	.831	670	1 14	.909	.418	2 18
05	050	.999	050	**.60**	565	825	.684	**1 15**	913	.408	2 23
.06	.060	998	060	61	.573	.820	699	1 16	.917	.399	2 30
.07	070	.998	070	.62	581	814	714	1 17	921	390	2 36
.08	080	.997	080	63	589	.808	.729	1 18	.925	.381	2 43
09	090	996	090	64	.597	802	.745	1 19	.928	.372	2 50
.10	100	.995	.100	**65**	.605	.796	.760	**1.20**	932	362	2 57
.11	110	.994	.110	66	.613	.790	.776	1 21	936	353	2 65
.12	120	.993	.121	.67	.621	784	792	1 22	939	344	2 73
.13	130	992	.131	.68	629	778	809	1 23	942	334	2 82
.14	140	.990	.141	69	.637	.771	.825	1 24	946	.325	2 91
15	.149	989	.151	**70**	644	.765	842	**1 25**	949	315	3 01
16	159	987	.161	71	652	.758	860	1 26	952	.306	3 11
17	169	986	.172	72	659	752	.877	1 27	.955	.296	3 22
18	179	.984	.182	73	.667	745	.895	1 28	958	287	3 34
19	189	982	.192	74	.674	738	.913	1 29	961	.277	3 47
20	199	.980	203	**75**	.682	.732	.932	**1 30**	964	268	3 60
21	208	978	.213	.76	.689	725	951	1 31	966	258	3 75
22	.218	.976	.224	.77	.696	718	970	1 32	969	248	3 90
23	228	.974	.234	.78	.703	.711	989	1 33	971	.238	4 07
24	238	.971	.245	.79	.710	.704	1 01	1 34	.973	.229	4 26
25	247	969	255	**80**	717	.697	1 03	**1 35**	976	.219	4 46
26	257	.966	.266	.81	724	690	1 05	1 36	978	.209	4 67
27	267	.964	.277	.82	731	682	1 07	1 37	980	199	4 91
28	.276	.961	.288	.83	738	.675	1 09	1 38	982	190	5 18
29	286	.958	.298	84	745	.667	1 12	1 39	984	180	5 47
.30	.296	.955	.309	**.85**	751	660	1 14	**1 40**	985	170	5 80
.31	305	.952	.320	.86	.758	652	1 16	1 41	987	.160	6 17
32	315	.949	.331	.87	764	645	1.19	1 42	989	150	6 58
33	324	946	343	.88	771	637	1 21	1 43	990	140	7 06
.34	.333	.943	354	.89	777	.629	1 23	1 44	.991	.130	7.60
35	343	.939	365	**.90**	783	622	1 26	**1 45**	.993	121	8 24
36	352	.936	376	91	790	.614	1 29	1 46	.994	111	8 96
37	362	932	388	92	796	.606	1 31	1 47	995	101	9 89
.38	.371	929	.399	.93	802	.598	1 34	1 48	.996	091	11 0
39	380	.925	411	94	.808	590	1 37	1 49	997	.081	12.4
40	389	.921	.423	**95**	813	582	1 40	**1 50**	.997	071	14 1
41	399	917	.435	96	.819	574	1 43	1 51	998	061	16 4
42	.408	913	447	.97	825	565	1.46	1 52	.999	.051	19 7
.43	.417	.909	.459	98	.831	.557	1 49	1 53	.999	.041	24 5
.44	426	905	.471	99	.836	.549	1 52	1 54	1 000	.031	32.5
45	435	900	483	**1.00**	.841	540	1 56	**1 55**	1 000	021	48 1
.46	444	.896	.495	1.01	847	532	1 59	1 56	1 000	011	92 6
47	453	892	.508	1.02	852	.523	1 63	1 57	1 000	.001	125 6
.48	.462	.887	521	1 03	.857	515	1 67	1 58	1.000	— 009	−108 7
.49	471	.882	.533	1.04	.862	.506	1 70	1 59	1 000	— 019	−52 1
.50	.479	878	546	**1 05**	867	498	1 74	**1 60**	1 000	— 029	−34 2
51	488	.873	.559	1.06	.872	.489	1 78	1 61	999	—.039	−25 5
52	.497	.868	.573	1 07	877	480	1 83	1.62	.999	— 049	−20 3
.53	.506	.863	586	1 08	.882	.471	1 87	1 63	998	— 059	−16 9
.54	.514	.858	.599	1.09	887	.462	1 92	1 64	998	— 069	−14.4

Table II — Degrees, Minutes, and Seconds to Radians

Degrees						Minutes		Seconds	
0°	0.00000 00	60°	1.04719 76	120°	2.09439 51	0′	0.00000 00	0″	0.00000 00
1	0.01745 33	61	1.06465 08	121	2.11184 84	1	0.00029 09	1	0.00000 48
2	0.03490 66	62	1.08210 41	122	2.12930 17	2	0.00058 18	2	0.00000 97
3	0.05235 99	63	1.09955 74	123	2.14675 50	3	0.00087 27	3	0.00001 45
4	0.06981 32	64	1.11701 07	124	2.16420 83	4	0.00116 36	4	0.00001 94
5	0.08726 65	**65**	1.13446 40	**125**	2.18166 16	**5**	0.00145 44	**5**	0.00002 42
6	0.10471 98	66	1.15191 73	126	2.19911 49	6	0.00174 53	6	0.00002 91
7	0.12217 30	67	1.16937 06	127	2.21656 82	7	0.00203 62	7	0.00003 39
8	0.13962 63	68	1.18682 39	128	2.23402 14	8	0 00232 71	8	0.00003 88
9	0.15707 96	69	1.20427 72	129	2.25147 47	9	0.00261 80	9	0.00004 36
10	0.17453 29	**70**	1.22173 05	**130**	2.26892 80	**10**	0.00290 89	**10**	0.00004 85
11	0.19198 62	71	1.23918 38	131	2.28638 13	11	0.00319 98	11	0.00005 33
12	0.20943 95	72	1.25663 71	132	2.30383 46	12	0.00349 07	12	0.00005 82
13	0.22689 28	73	1.27409 04	133	2 32128 79	13	0.00378 15	13	0 00006 30
14	0.24434 61	74	1.29154 36	134	2.33874 12	14	0.00407 24	14	0.00006 79
15	0.26179 94	**75**	1.30899 69	**135**	2.35619 45	**15**	0.00436 33	**15**	0.00007 27
16	0.27925 27	76	1.32645 02	136	2.37364 78	16	0.00465 42	16	0.00007 76
17	0.29670 60	77	1.34390 35	137	2.39110 11	17	0 00494 51	17	0.00008 24
18	0.31415 93	78	1.36135 68	138	2.40855 44	18	0.00523 60	18	0.00008 73
19	0.33161 26	79	1.37881 01	139	2.42600 77	19	0.00552 69	19	0.00009 21
20	0.34906 59	**80**	1.39626 34	**140**	2.44346 10	**20**	0.00581 78	**20**	0.00009 70
21	0.36651 91	81	1.41371 67	141	2.46091 42	21	0.00610 87	21	0.00010 18
22	0.38397 24	82	1.43117 00	142	2.47836 75	22	0.00639 95	22	0.00010 67
23	0.40142 57	83	1.44862 33	143	2.49582 08	23	0.00669 04	23	0.00011 15
24	0.41887 90	84	1.46607 66	144	2.51327 41	24	0.00698 13	24	0.00011 64
25	0.43633 23	**85**	1.48352 99	**145**	2.53072 74	**25**	0.00727 22	**25**	0.00012 12
26	0.45378 56	86	1.50098 32	146	2.54818 07	26	0.00756 31	26	0.00012 61
27	0.47123 89	87	1.51843 64	147	2.56563 40	27	0.00785 40	27	0.00013 09
28	0.48869 22	88	1.53588 97	148	2.58308 73	28	0.00814 49	28	0.00013 57
29	0.50614 55	89	1.55334 30	149	2.60054 06	29	0.00843 58	29	0.00014 06
30	0.52359 88	**90**	1.57079 63	**150**	2.61799 39	**30**	0.00872 66	**30**	0.00014 54
31	0.54105 21	91	1.58824 96	151	2.63544 72	31	0.00901 75	31	0.00015 03
32	0.55850 54	92	1.60570 29	152	2.65290 05	32	0.00930 84	32	0.00015 51
33	0.57595 87	93	1.62315 62	153	2.67035 38	33	0.00959 93	33	0.00016 00
34	0.59341 19	94	1.64060 95	154	2.68780 70	34	0.00989 02	34	0.00016 48
35	0.61086 52	**95**	1.65806 28	**155**	2.70526 03	**35**	0.01018 11	**35**	0.00016 97
36	0.62831 85	96	1.67551 61	156	2.72271 36	36	0.01047 20	36	0.00017 45
37	0.64577 18	97	1.69296 94	157	2.74016 69	37	0.01076 29	37	0.00017 94
38	0.66322 51	98	1.71042 27	158	2.75762 02	38	0.01105 38	38	0.00018 42
39	0.68067 84	99	1.72787 60	159	2.77507 35	39	0.01134 46	39	0.00018 91
40	0.69813 17	**100**	1.74532 93	**160**	2.79252 68	**40**	0.01163 55	**40**	0.00019 39
41	0.71558 50	101	1.76278 25	161	2.80998 01	41	0.01192 64	41	0.00019 88
42	0.73303 83	102	1.78023 58	162	2.82743 34	42	0.01221 73	42	0.00020 36
43	0.75049 16	103	1.79768 91	163	2.84488 67	43	0.01250 82	43	0.00020 85
44	0.76794 49	104	1.81514 24	164	2.86234 00	44	0.01279 91	44	0.00021 33
45	0.78539 82	**105**	1.83259 57	**165**	2.87979 33	**45**	0.01309 00	**45**	0.00021 82
46	0.80285 15	106	1.85004 90	166	2.89724 66	46	0.01338 09	46	0.00022 30
47	0.82030 47	107	1.86750 23	167	2.91469 99	47	0.01367 17	47	0.00022 79
48	0.83775 80	108	1.88495 56	168	2.93215 31	48	0.01396 26	48	0.00023 27
49	0.85521 13	109	1.90240 89	169	2.94960 64	49	0.01425 35	49	0.00023 76
50	0.87266 46	**110**	1.91986 22	**170**	2.96705 97	**50**	0.01454 44	**50**	0.00024 24
51	0.89011 79	111	1.93731 55	171	2.98451 30	51	0.01483 53	51	0.00024 73
52	0.90757 12	112	1.95476 88	172	3.00196 63	52	0.01512 62	52	0.00025 21
53	0.92502 45	113	1.97222 21	173	3.01941 96	53	0.01541 71	53	0.00025 70
54	0.94247 78	114	1.98967 53	174	3.03687 29	54	0.01570 80	54	0.00026 18
55	0.95993 11	**115**	2.00712 86	**175**	3.05432 62	**55**	0.01599 89	**55**	0.00026 66
56	0.97738 44	116	2.02458 19	176	3.07177 95	56	0.01628 97	56	0.00027 15
57	0.99483 77	117	2.04203 52	177	3.08923 28	57	0.01658 06	57	0.00027 63
58	1.01229 10	118	2.05948 85	178	3.10668 61	58	0.01687 15	58	0.00028 12
59	1.02974 43	119	2.07694 18	179	3.12413 94	59	0.01716 24	59	0.00028 60

Table III — Radians to Degrees

	Radians	Tenths	Hundredths	Thousandths	Ten-Thousandths
	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″
1	57 17 44.8	5 43 46.5	0 34 22.6	0 3 26.3	0 0 20.6
2	114 35 29.6	11 27 33.0	1 08 45.3	0 6 52.5	0 0 41.3
3	171 53 14.4	17 11 19.4	1 43 7.9	0 10 18.8	0 1 01.9
4	229 10 59.2	22 55 05.9	2 17 30.6	0 13 45.1	0 1 22.5
5	286 28 44.0	28 38 52.4	2 51 53.2	0 17 11.3	0 1 43.1
6	343 46 28.8	34 22 38.9	3 26 15.9	0 20 37.6	0 2 03.8
7	401 04 13.6	40 06 25.4	4 00 38.5	0 24 03.9	0 2 24.4
8	458 21 58.4	45 50 11.8	4 35 1.2	0 27 30.1	0 2 45.0
9	515 39 43.3	51 33 58.3	5 09 23.8	0 30 56.4	0 3 05.6

ANSWERS

[Answers are given to the odd-numbered problems only.]

CHAPTER I

Page 4 (§ 2)

1. (a) -8; (b) -3; (c) 11; (d) 5; (e) 3; (f) -8.

Page 7 (§ 3)

7. (a) $y_1 - y_4$; (b) $x_2 - x_3$; (c) $y_1 - y_2$; (d) $x_3 - x_4$; (e) $y_1 - y_3$; (f) $x_2 - x_4$.

Page 9 (§ 4)

21. 669°, 1029°, 1389°, −51°, −411°, −771°.

23. 204° 30′, 564° 30′, 924° 30′, −515° 30′, −875° 30′, −1235° 30′.

25. 16°, 376°, 736°, −704°, −1064°, −1424°.

27. 507° 45′ 33″, 867° 45′ 33″, 1227° 45′ 33″, −212° 14′ 27″, −572° 14′ 27″, −932° 14′ 27″.

Page 13 (§ 5)

1. II; IV; I; III.

3. II; II; II; II.

5. $-\dfrac{11\pi}{6}$.

7. $\dfrac{5\pi}{2}$.

9. $-\dfrac{13\pi}{4}$.

11. $\dfrac{31\pi}{6}$.

13. $-\dfrac{31061\pi}{18000}$.

15. 2.90580.

17. 5.34263.

19. 6.58434.

21. 0.00069.

23. 210°.

25. −257° 49′ 51″.6.

27. 360° 33′ 44″.4.

29. 130° 55′ 5″.9.

31. 5′ 30″.1.

33. −90° 0′ 0″.7.

35. $\frac{17\pi}{8}, \frac{33\pi}{8}, \frac{49\pi}{8}, -\frac{15\pi}{8}, -\frac{31\pi}{8}, -\frac{47\pi}{8}$.

37. $\frac{\pi}{3}, \frac{13\pi}{3}, \frac{19\pi}{3}, -\frac{5\pi}{3}, -\frac{11\pi}{3}, -\frac{17\pi}{3}$.

39. $\frac{7\pi}{4}, \frac{15\pi}{4}, \frac{23\pi}{4}, -\frac{9\pi}{4}, -\frac{17\pi}{4}, -\frac{25\pi}{4}$.

41. $\frac{13\pi}{5}, \frac{23\pi}{5}, \frac{33\pi}{5}, -\frac{7\pi}{5}, -\frac{17\pi}{5}, -\frac{27\pi}{5}$.

43. 9.34, 15.62, 21.90, −3.22, −9.50, −15.78.

45. 5.141, 11.424, 17.707, −7.425, −13.708, −19.991.

Page 14 (§ 5)

47. $154° 13' 0''.9$. **51.** 1.5655 ft. **55.** 42.955 in.
49. 6.1761 cm. **53.** $137° 30' 35''.5$.

Page 17 (§ 6)

3. $+, -, -, -, -, +$. **9.** $-, -, +, +, -, -$.
5. $-, -, +, +, -, -$. **11.** $+, -, -, -, -, +$.
7. $+, -, -, -, -, +$. **13.** $+, -, -, -, -, +$.
15. II, IV. **19.** II, III. **23.** II.
17. I, IV. **21.** III, IV. **25.** III.

Page 20 (§ 7)

1. $-\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2}, -1, -1, \sqrt{2}, -\sqrt{2}$.
3. $-\frac{1}{2}, -\frac{1}{2}\sqrt{3}, \frac{1}{3}\sqrt{3}, \sqrt{3}, -\frac{2}{3}\sqrt{3}, -2$.
5. $\frac{1}{2}, \frac{1}{2}\sqrt{3}, \frac{1}{3}\sqrt{3}, \sqrt{3}, \frac{2}{3}\sqrt{3}, 2$.
7. $\frac{1}{2}\sqrt{2}, -\frac{1}{2}\sqrt{2}, -1, -1, -\sqrt{2}, \sqrt{2}$.
9. $-\frac{1}{2}\sqrt{3}, -\frac{1}{2}, \sqrt{3}, \frac{1}{3}\sqrt{3}, -2, -\frac{2}{3}\sqrt{3}$.
11. 135°, 315°. **15.** 150°, 330°. **19.** 120°, 240°.
13. 60°, 300°. **17.** 240°, 300°. **21.** 210°, 330°.

Page 21 (§ 7)

23. $\frac{1}{2}(1+\sqrt{3})$. **25.** $-\frac{1}{3}(3+\sqrt{3})$. **27.** $-\frac{1}{4}$.

Page 23 (§ 8)

1. III: $\sin\theta_3 = -\frac{5}{13}$; $\cos\theta_3 = -\frac{12}{13}$; $\tan\theta_3 = \frac{5}{12}$;
$\operatorname{ctn}\theta_3 = \frac{12}{5}$; $\sec\theta_3 = -\frac{13}{12}$.

IV: $\sin\theta_4 = -\frac{5}{13}$; $\cos\theta_4 = \frac{12}{13}$; $\tan\theta_4 = -\frac{5}{12}$;
$\operatorname{ctn}\theta_4 = -\frac{12}{5}$; $\sec\theta_4 = \frac{13}{12}$.

3. II: $\sin\theta_2 = \frac{2}{13}\sqrt{13}$; $\cos\theta_2 = -\frac{3}{13}\sqrt{13}$;
$\operatorname{ctn}\theta_2 = -\frac{3}{2}$; $\sec\theta_2 = -\frac{1}{3}\sqrt{13}$; $\csc\theta_2 = \frac{1}{2}\sqrt{13}$.

IV: $\sin\theta_4 = -\frac{2}{13}\sqrt{13}$; $\cos\theta_4 = \frac{3}{13}\sqrt{13}$; $\operatorname{ctn}\theta_4 = -\frac{3}{2}$;
$\sec\theta_4 = \frac{1}{3}\sqrt{13}$; $\csc\theta_4 = -\frac{1}{2}\sqrt{13}$.

5. I: $\cos\theta_1 = \frac{1}{7}\sqrt{33}$; $\tan\theta_1 = \frac{4}{33}\sqrt{33}$; $\operatorname{ctn}\theta_1 = \frac{1}{4}\sqrt{33}$;
$\sec\theta_1 = \frac{7}{33}\sqrt{33}$; $\csc\theta_1 = \frac{7}{4}$.

II: $\cos\theta_2 = -\frac{1}{7}\sqrt{33}$; $\tan\theta_2 = -\frac{4}{33}\sqrt{33}$;
$\operatorname{ctn}\theta_2 = -\frac{1}{4}\sqrt{33}$; $\sec\theta_2 = -\frac{7}{33}\sqrt{33}$; $\csc\theta_2 = \frac{7}{4}$.

7. II: $\sin\theta_2 = \frac{60}{61}$; $\cos\theta_2 = -\frac{11}{61}$; $\operatorname{ctn}\theta_2 = -\frac{11}{60}$;
$\sec\theta_2 = -\frac{61}{11}$; $\csc\theta_2 = \frac{61}{60}$.

9. IV: $\sin\theta_4 = -\frac{7}{25}$; $\cos\theta_4 = \frac{24}{25}$; $\tan\theta_4 = -\frac{7}{24}$;
$\operatorname{ctn}\theta_4 = -\frac{24}{7}$; $\sec\theta_4 = \frac{25}{24}$; $\csc\theta_4 = -\frac{25}{7}$.

11. IV: $\cos\theta_4 = \frac{9}{41}$; $\tan\theta_4 = -\frac{40}{9}$; $\operatorname{ctn}\theta_4 = -\frac{9}{40}$;
$\sec\theta_4 = \frac{41}{9}$; $\csc\theta_4 = -\frac{41}{40}$.

13. $\frac{16}{111}$. **15.** $\frac{2}{3}\sqrt{5}$.

Page 26 (§ 9)

3. IV: $\sin\theta_4 = -\frac{5}{13}$; $\cos\theta_4 = \frac{12}{13}$; $\tan\theta_4 = -\frac{5}{12}$;
$\sec\theta_4 = \frac{13}{12}$; $\csc\theta_4 = -\frac{13}{5}$.

5. III: $\sin\theta_3 = -\frac{24}{25}$; $\cos\theta_3 = -\frac{7}{25}$; $\operatorname{ctn}\theta_3 = \frac{7}{24}$;
$\sec\theta_3 = -\frac{25}{7}$; $\csc\theta_3 = -\frac{25}{24}$.

7. IV: $\sin\theta_4 = -\frac{8}{17}$; $\cos\theta_4 = \frac{15}{17}$; $\tan\theta_4 = -\frac{8}{15}$;
$\operatorname{ctn}\theta_4 = -\frac{15}{8}$; $\sec\theta_4 = \frac{17}{15}$.

Page 27 (§ 10)

1. $\sin\theta = \sqrt{1-\cos^2\theta}$; $\tan\theta = \dfrac{\sqrt{1-\cos^2\theta}}{\cos\theta}$;

$\operatorname{ctn}\theta = \dfrac{\cos\theta}{\sqrt{1-\cos^2\theta}}$; $\sec\theta = \dfrac{1}{\cos\theta}$; $\csc\theta = \dfrac{1}{\sqrt{1-\cos^2\theta}}$.

3. $\sin\theta = \dfrac{1}{\csc\theta}$; $\cos\theta = \dfrac{\sqrt{\csc^2\theta-1}}{\csc\theta}$; $\tan\theta = \dfrac{1}{\sqrt{\csc^2\theta-1}}$;

$\operatorname{ctn}\theta = \sqrt{\csc^2\theta-1}$; $\sec\theta = \dfrac{\csc\theta}{\sqrt{\csc^2\theta-1}}$.

5. $\cos\phi = \sqrt{1-\sin^2\phi}$; $\tan\phi = \dfrac{\sin\phi}{\sqrt{1-\sin^2\phi}}$;

$\operatorname{ctn}\phi = \dfrac{\sqrt{1-\sin^2\phi}}{\sin\phi}$; $\sec\phi = \dfrac{1}{\sqrt{1-\sin^2\phi}}$; $\csc\phi = \dfrac{1}{\sin\phi}$.

7. $\sin\phi = \dfrac{\tan\phi}{\sqrt{1+\tan^2\phi}}$; $\cos\phi = \dfrac{1}{\sqrt{1+\tan^2\phi}}$;

$\operatorname{ctn}\phi = \dfrac{1}{\tan\phi}$; $\sec\phi = \sqrt{1+\tan^2\phi}$; $\csc\phi = \dfrac{\sqrt{1+\tan^2\phi}}{\tan\phi}$.

Page 28 (§ 10)

9. $$\operatorname{ctn}\theta = \frac{\sqrt{1-\sin^2\theta}}{\sin\theta} = \frac{\cos\theta}{\sqrt{1-\cos^2\theta}} = \frac{1}{\tan\theta}$$

$$= \frac{1}{\sqrt{\sec^2\theta-1}} = \sqrt{\csc^2\theta-1}.$$

11. $$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{\sqrt{1-\cos^2\theta}} = \frac{\sqrt{1+\tan^2\theta}}{\tan\theta}$$

$$= \sqrt{1+\operatorname{ctn}^2\theta} = \frac{\sec\theta}{\sqrt{\sec^2\theta-1}}.$$

13. $$\sec\theta = \frac{1}{\sqrt{1-\sin^2\theta}} = \frac{1}{\cos\theta} = \sqrt{1+\tan^2\theta}$$

$$= \frac{\sqrt{1+\operatorname{ctn}^2\theta}}{\operatorname{ctn}\theta} = \frac{\csc\theta}{\sqrt{\csc^2\theta-1}}.$$

Page 31 (**General Exercises**)

1. $-1297° \, 12'.0$. **5.** $-\frac{1}{4}$.

7. $\sin\beta = \dfrac{1}{\sqrt{1+\operatorname{ctn}^2\beta}}$; $\cos\beta = \dfrac{\operatorname{ctn}\beta}{\sqrt{1+\operatorname{ctn}^2\beta}}$;

$\tan\beta = \dfrac{1}{\operatorname{ctn}\beta}$; $\sec\beta = \dfrac{\sqrt{1+\operatorname{ctn}^2\beta}}{\operatorname{ctn}\beta}$; $\csc\beta = \sqrt{1+\operatorname{ctn}^2\beta}$.

9. (a) $399° \, 48'.7$, $759° \, 48'.7$, $1119° \, 48'.7$, $-320° \, 11'.3$, $-680° \, 11'.3$, $-1040° \, 11'.3$.

(b) $\dfrac{18\pi}{5}$, $\dfrac{28\pi}{5}$, $\dfrac{38\pi}{5}$, $-\dfrac{2\pi}{5}$, $-\dfrac{12\pi}{5}$, $-\dfrac{22\pi}{5}$.

(c) 2.56, 8.84, 15.12, -10.00, -16.28, -22.56.

(d) $115° \, 4' \, 28''$, $475° \, 4' \, 28''$, $835° \, 4' \, 28''$, $-604° \, 55' \, 32''$, $-964° \, 55' \, 32''$, $-1324° \, 55' \, 32''$.

Page 32 (**General Exercises**)

17. 0.81877 ft.

19. (a) 120°, 300°; (b) 30°, 330°; (c) 225°, 315°; (d) 135°, 225°; (e) 60°, 240°; (f) 30°, 150°.

23. 0.00015 rad. per sec.; 0.00175 rad. per sec.; 0.10472 rad. per sec.

27. $\frac{1}{6}\sqrt{3}$.

Page 33 (**General Exercises**)

31. IV: $\sin B = -\frac{40}{41}$; $\cos B = \frac{9}{41}$; $\tan B = -\frac{40}{9}$; $\operatorname{ctn} B = -\frac{9}{40}$; $\sec B = \frac{41}{9}$.

33. $63° \, 39' \, 20''.2$. **35.** $\sin^2\theta$. **37.** 365,160 ft. **43.** 30°, 330°.

Page 34 (**General Exercises**)

45. 240°, 300°.

CHAPTER II

Page 37 (§ 13)

1. cos $22° 47'.6$. **5.** sec 0.28. **9.** $40°$.

3. ctn $6° 23' 17''$. **7.** $50°$.

Page 40 (§ 14)

1. 0.41199, 0.91118, 0.45215, 2.2116.

3. 0.55805, 0.82981, 0.67249, 1.4870.

5. 0.69136, 0.72251, 0.95690, 1.0450.

7. $16° 25'.4$. **11.** $59° 4'.2$. **15.** $54° 38'.8$. **19.** 0.29859.

9. $42° 56'.4$. **13.** $10° 22'.3$. **17.** $9.79346-10$.

Page 41 (§ 14)

21. 0.00515. **31.** $81° 44'.9$. **41.** -22.205.

23. $9.99581-10$. **33.** $31° 8'.9$. **43.** 2.1103.

25. $9.60302-10$. **35.** $67° 27'.9$. **45.** 1.0363.

27. $17° 29'.5$. **37.** $60° 0'.1$.

29. $65° 47'.3$. **39.** $45° 1'.9$.

Page 44 (§ 15)

1. $B = 46° 12' 44''$, $a = 2.6286$, $b = 2.7422$, $K = 3.6041$.

3. $A = 25° 49' 21''$, $a = 42.811$, $c = 98.283$.

Page 45 (§ 15)

5. $B = 63° 1'.7$, $a = 0.45806$, $c = 1.0100$.

7. $A = 28° 6'.4$, $B = 61° 53'.6$, $a = 385.05$ or 385.06, $K = 138{,}800$.

9. $B = 52° 51' 16''$, $b = 8.0400$, $c = 10.087$.

11. $A = 30° 13'.4$, $B = 59° 46'.6$, $a = 22.831$.

13. $A = 48° 1'.2$, $a = 0.94165$, $c = 1.2667$, $K = 0.39892$.

15. $A = 10° 33'.4$, $B = 79° 26'.6$, $b = 7.6878$ or 7.6880.

17. $B = 11° 53'.7$
$a = 956.10$
$c = 977.06.$

19. $B = 42° 26'.4$
$a = 1.6017$
$b = 1.4646.$

21. $A = 21° 48'.1$
$B = 68° 11'.9$
$c = 18.973$
$K = 62.065.$

23. $A = 24° 26'.8$
$B = 65° 33'.2$
$a = 6000.7$
or 6000.8.

25. $A = 34° 51'.4$
$B = 110° 17'.2$
$b = 44.470.$

27. $B = 66° 20'.4$
$b = 110.45$
$h = 84.490.$

29. $A = 66° 13'.6$
$B = 47° 32'.8$
$h = 16.126.$

31. $A = 39° 35'.1$
$a = 6.7512$
$h = 4.3020.$

33. 13.671 in.

35. 200.05 sq. cm.; 52.650 cm.

37. 1.4489.

39. 133.93.

Page 47 (§ 16)

1. $C = 50° 0'.1$
$a = 3.7396$
$b = 4.4263.$

3. $A = 28° 16'.1$
$B = 82° 2'.6$
$b = 378.44.$

5. $A = 40° 13'.5$
$B = 87° 47'.2$
$C = 51° 59'.3.$

7. $A = 47° 23'.9$
$C = 49° 5'.6$
$a = 1091.7.$

9. $B = 91° 52' 19''$
$C = 29° 40' 52''$
$b = 28.228.$

11. $B = 45° 51'.8$
$C = 57° 7'.6$
$a = 0.12812.$

Page 53 (§ 18)

1. 40° 33'.2.

3. 27.031 yds.

5. 29° 11'.6.

7. 8° 0'.0.

9. 14° 53'.8; 7.0200 in.

Page 54 (§ 18)

11. 237.86 ft.

13. 200.00 ft.

15. 120.55 ft.

17. 116.26 ft.

19. 43.010 ft.

Page 55 (§ 18)

21. 122.47 ft. **23.** 802.18 ft. **25.** 50.479 ft. **27.** 44.818 ft.

Page 55 (General Exercises)

3. $B = 51° 23'.1$
$a = 12.130$
$c = 13.340.$

5. 44° 36'.0.

Page 56 (General Exercises)

7. 1975.5 ft.

9. $\frac{4\pi}{7}$.

11. 254.38 ft.

13. 68.371 in.

15. 0.86952.

Page 57 (General Exercises)

19. $35° 15'.8$.

21. 91.452 ft.

23. $31° 42'.9$.

25. 327.15 cu. in.;
198.87 sq. in.

27. 104.06 ft.

Page 58 (General Exercises)

31. $A = 51° 18'.9$
$C = 79° 24'.2$
$b = 6.7588$.

35. 255.49 yds.

37. 1.1520 cm.

Page 59 (General Exercises)

39. 13.476 in.

CHAPTER III

Page 62 (§ 20)

1. $\sin(-\theta) = -\sin\theta$; $\cos(-\theta) = \cos\theta$;
$\tan(-\theta) = -\tan\theta$.

5. $\sin(-217°) = -\sin 217°$; $\cos(-217°) = \cos 217°$;
$\tan(-217°) = -\tan 217°$; $\text{ctn}(-217°) = -\text{ctn}\, 217°$;
$\sec(-217°) = \sec 217°$; $\csc(-217°) = -\csc 217°$.

7. $\sin(-193° 18' 16'') = -\sin 193° 18' 16''$;
$\cos(-193° 18' 16'') = \cos 193° 18' 16''$;
$\tan(-193° 18' 16'') = -\tan 193° 18' 16''$;
$\text{ctn}(-193° 18' 16'') = -\text{ctn}\, 193° 18' 16''$;
$\sec(-193° 18' 16'') = \sec 193° 18' 16''$;
$\csc(-193° 18' 16'') = -\csc 193° 18' 16''$.

9. $\sin(-0.769) = -\sin 0.769$; $\cos(-0.769) = \cos 0.769$;
$\tan(-0.769) = -\tan 0.769$; $\text{ctn}(-0.769) = -\text{ctn}\, 0.769$;
$\sec(-0.769) = \sec 0.769$; $\csc(-0.769) = -\csc 0.769$.

11. -0.19920, 0.97996, -0.20327, -4.9196.

13. -0.37922, 0.92531, -0.40982, -2.4401.

15. -0.93177, 0.36304, -2.5666, -0.38963.

17. $\cos B - \sin B$.

Page 63 (§ 20)

19. $\frac{3}{5}$, $-\frac{4}{5}$, $-\frac{3}{4}$.

Page 65 (§ 22)

1. $\sin(90° - \theta) = \cos\theta$; $\cos(90° - \theta) = \sin\theta$;
$\tan(90° - \theta) = \text{ctn}\,\theta$.

3. $\sin(90° + \theta) = \cos\theta$; $\cos(90° + \theta) = -\sin\theta$;
$\tan(90° + \theta) = -\text{ctn}\,\theta$.

Page 66 (§ 22)

9. 0.21650, -0.97629, -0.22175, -4.5095.
11. 0.73860, -0.67415, -1.0956, -0.91274.
13. -0.13203, -0.99124, 0.13319, 7.5080.
15. 0.38268, -0.92388, -0.41421, -2.4142.
17. -0.38268, -0.92388, 0.41421, 2.4142.
19. $\csc^2 B$.

Page 67 (§ 23)

1. 0.87890, -0.47700, -1.8425.
3. 0.25634, -0.96388, -0.27632.
5. 0.35282, -0.93569, -0.37707.
7. 2.92189.

Page 68 (§ 24)

1. $\sin(90° + \theta) = \cos\theta$; $\cos(90° + \theta) = -\sin\theta$;
$\tan(90° + \theta) = -\text{ctn}\,\theta$.

3. $\sin(180° + \theta) = -\sin\theta$; $\cos(180° + \theta) = -\cos\theta$;
$\tan(180° + \theta) = \tan\theta$.

5. $\sin(270° - \theta) = -\cos\theta$; $\cos(270° - \theta) = -\sin\theta$;
$\tan(270° - \theta) = \text{ctn}\,\theta$.

7. $\sin\left(\frac{3\pi}{2} + \theta\right) = -\cos\theta$; $\cos\left(\frac{3\pi}{2} + \theta\right) = \sin\theta$;

$\tan\left(\frac{3\pi}{2} + \theta\right) = -\text{ctn}\,\theta$.

Page 69 (§ 24)

9. $\sin(\theta - 270°) = \cos\theta$; $\cos(\theta - 270°) = -\sin\theta$; $\tan(\theta - 270°) = -\text{ctn}\,\theta$.

11. $\sin(-180° - \theta) = \sin\theta$; $\cos(-180° - \theta) = -\cos\theta$; $\tan(-180° - \theta) = -\tan\theta$.

13. $\sin(-360° - \theta) = -\sin\theta$; $\cos(-360° - \theta) = \cos\theta$; $\tan(-360° - \theta) = -\tan\theta$.

15. $\sin(450° + \theta) = \cos\theta$; $\cos(450° + \theta) = -\sin\theta$; $\tan(450° + \theta) = -\text{ctn}\,\theta$.

Page 71 (§ 25)

1. $-\sin\theta$.
3. $\cos\alpha$.
5. $-\cos\theta$.
7. $-\cos\beta$.
9. $\text{ctn}\,\theta$.
11. $-\tan\phi$.
13. $-\sin\theta$.
15. $-\sin\beta$.
17. $\cos\theta$.
19. $-\sec\beta$.
21. $\sec A$.
23. $\sin^2\theta + \cos\theta$.
25. -1.
27. 0.15221, −0.98835, −0.15400, −6.4933.
29. −0.66432, 0.74745, −0.88878, −1.1251.
31. 0.949, −0.315, −3.01, −0.332.
33. 0.95469, −0.29758, −3.2082, −0.31170.

Page 76 (§ 27)

1. 0, −1, 0.
3. 0, 1, 0.

Page 77 (§ 27)

5. −1.
7. 0.

Page 78 (§ 28)

1.

As the angle varies from	$0 \to \frac{\pi}{2}$	$\frac{\pi}{2} \to \pi$	$\pi \to \frac{3\pi}{2}$	$\frac{3\pi}{2} \to 2\pi$
its cosine varies from	$1 \to 0$	$0 \to -1$	$-1 \to 0$	$0 \to 1$

3.

As θ varies from	$0 \to \frac{\pi}{4}$	$\frac{\pi}{4} \to \frac{\pi}{2}$	$\frac{\pi}{2} \to \frac{3\pi}{4}$	$\frac{3\pi}{4} \to \pi$	$\pi \to \frac{5\pi}{4}$	$\frac{5\pi}{4} \to \frac{3\pi}{2}$
its sine varies from	$0 \to \frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2} \to 1$	1 to $\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2} \to 0$	$0 \to -\frac{1}{2}\sqrt{2}$	$-\frac{1}{2}\sqrt{2} \to -1$
its cosine varies from	$1 \to \frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2} \to 0$	$0 \to -\frac{1}{2}\sqrt{2}$	$-\frac{1}{2}\sqrt{2} \to -1$	$-1 \to -\frac{1}{2}\sqrt{2}$	$-\frac{1}{2}\sqrt{2} \to 0$
its tangent varies from	$0 \to 1$	$1 \to +\infty$	$-\infty \to -1$	$-1 \to 0$	$0 \to 1$	$1 \to +\infty$

Page 79 (§ 29)

1. 2π. **3.** 3. **5.** π.

Page 80 (§ 29)

7. $\frac{\pi}{2}$. **9.** 2. **11.** 3π. **13.** $\frac{2\pi}{3}$. **15.** 2π.

17. The periods are 2π and π respectively.

19. The period of each is $\frac{2\pi}{b}$. **21.** 1. **23.** π.

Page 82 (§ 30)

1. 60°, 120°.

3. 210°, 330°.

5. 60°, 300°.

7. 128° 11′.1, 308° 11′.1.

9. 225°, 315°.

11. 4° 32′.9, 184° 32′.9.

13. $\frac{\pi}{4}, \frac{5\pi}{4}$.

15. $\frac{5\pi}{6}, \frac{7\pi}{6}$.

17. π.

19. $\frac{\pi}{2}, \frac{3\pi}{2}$.

21. $\frac{\pi}{2}, \frac{3\pi}{2}$.

23. $\frac{\pi}{3}$ or $\frac{2\pi}{3}$.

25. $\frac{3\pi}{4}$ or $\frac{7\pi}{4}$.

27. $-\frac{5}{18}$.

29. $\text{arc cos}\,\frac{3x^2}{4}$, $\text{arc tan}\,\frac{\sqrt{16 - 9x^4}}{3x^2}$, etc.

31. $\text{arc sin}\,\frac{e^{-x}}{\sqrt{e^{2x} + e^{-2x}}}$, $\text{arc tan}\, e^{-2x}$, etc.

Page 84 (§ 31)

1. 270°.
3. 180°.
5. 90°, 270°.
7. 184° 29′.0, 355° 31′.0.
9. 243° 47′.4, 296° 12′.6.
11. 89° 7′.7, 270° 52′.3.
13. 0°, 360°.
15. 135°, 315°.
17. 0°, 60°, 300°, 360°.
19. 0°, 135°, 180°, 225°, 360°.
21. 90°.
23. 14° 28′.6, 165° 31′.4.
25. 60°, 90°, 120°, 240°, 270°, 300°.
27. 192° 39′.8, 347° 20′.2.
29. No solution.
31. 67° 30′.0, 157° 30′.0, 247° 30′.0, 337° 30′.0.
33. 0°, 30°, 60°, 150°, 180°, 300°, 360°.

Page 85 (General Exercises)

1. $\sin(-\theta) = \frac{12}{13}$; $\cos(-\theta) = \frac{5}{13}$; $\tan(-\theta) = \frac{12}{5}$;
$\text{ctn}(-\theta) = \frac{5}{12}$; $\sec(-\theta) = \frac{13}{5}$; $\csc(-\theta) = \frac{13}{12}$.

3. $-1 - \cos^2 x$.
5. $-2 \sin C$.

7. $\sin(270° + \theta) = -\cos\theta$; $\cos(270° + \theta) = \sin\theta$;
$\tan(270° + \theta) = -\text{ctn}\,\theta$.

9. $\sin\theta_3 = \dfrac{e^{-x} - e^x}{e^x + e^{-x}}$; $\cos\theta_3 = -\dfrac{2}{e^x + e^{-x}}$; $\text{ctn}\,\theta_3 = \dfrac{2}{e^x - e^{-x}}$;
$\sec\theta_3 = -\dfrac{e^x + e^{-x}}{2}$; $\csc\theta_3 = \dfrac{e^x + e^{-x}}{e^{-x} - e^x}$.

11. $-\infty$.
13. $3 - 2\sqrt{3}$.

15. $\sin(-270° - \theta) = \cos\theta$; $\cos(-270° - \theta) = \sin\theta$;
$\tan(-270° - \theta) = \text{ctn}\,\theta$.

Page 86 (General Exercises)

17. True for $\theta = 45°$; conditional equation.
19. All values of θ.
23. 0.63292.
25. -0.24632.
27. 0.

Page 87 (General Exercises)

31. $-\frac{7}{8}$.
33. Yes.
37. $\dfrac{15\pi}{4}$.

Page 88 (General Exercises)

39. $\sec^2 B$.
41. -0.43411.
45. 69° 50′.7, 290° 9′.3.
47. 191° 50′.4, 348° 9′.6.
49. -0.37008.

CHAPTER IV

Page 93 (§ 34)

3. 1. **5.** $\frac{1}{4}(\sqrt{2}+\sqrt{6})$. **7.** $\frac{1}{4}(\sqrt{6}-\sqrt{2})$. **9.** $\frac{1}{4}(\sqrt{2}-\sqrt{6})$.

Page 94 (§ 34)

11. $-\frac{16}{65}$. **13.** $\frac{13}{85}$. **15.** $\sin 3A$. **17.** $2C + D$.

Page 95 (§ 36)

3. $-\frac{24}{25}$. **7.** $\frac{323}{325}$. **11.** $4A$. **17.** $\frac{1}{10}(3-4\sqrt{3})$.

5. $\frac{7}{25}$. **9.** $-\frac{253}{325}$. **13.** $-2A$.

Page 96 (§ 36)

19. $-\frac{33}{65}$. **21.** $3\sin x - 4\sin^3 x$.

Page 97 (§ 37)

3. $\frac{36}{77}$. **5.** $-\frac{84}{13}$.

7. (a) No; (b) yes; (c) no; (d) yes.

Page 98 (§ 37)

9. $\frac{1}{18}$. **13.** $\frac{7}{4}$. **17.** arc tan $(-\frac{15}{8})$.

11. 2 or $-\frac{2}{11}$. **15.** arc tan ∞.

Page 102 (§ 39)

1. $-\frac{336}{625}$. **7.** $\frac{240}{289}$. **13.** $-\frac{7}{25}$. **17.** A.

3. $\frac{336}{527}$. **9.** $-\frac{240}{161}$. **15.** $-\frac{1}{2}$. **19.** $2A$.

5. $\frac{3}{5}$. **11.** $\frac{5}{34}\sqrt{34}$.

Page 103 (§ 39)

21. $8A$. **23.** $\frac{24}{25}$. **25.** $\frac{1}{10}\sqrt{10}$.

27. $\frac{1}{2}\sqrt{2-\sqrt{2}}$; $\frac{1}{2}\sqrt{2+\sqrt{2}}$; $\sqrt{2}-1$.

29. $\frac{1}{2}\sqrt{2+\sqrt{3}}$; $-\frac{1}{2}\sqrt{2-\sqrt{3}}$; $-2-\sqrt{3}$.

31. $\frac{1}{2}\sqrt{2-2b}$.

33. $\frac{1}{2}\sqrt{2+2b}$.

35. $\frac{1-b}{a}$, or $\frac{a}{1+b}$.

Page 105 (§ 40)

3. $2 \cos 3\theta \sin \theta$.
5. $2 \cos 3\theta \cos \theta$.
7. $2 \cos \frac{5\theta}{2} \sin \frac{\theta}{2}$.
9. $-2 \cos \frac{5\theta}{2} \sin \frac{\theta}{2}$.
11. $2 \cos 4\theta \cos 2\theta$.
13. $-2 \cos 3\theta \sin \theta$.
15. $-2 \sin \frac{11\theta}{2} \sin \frac{5\theta}{2}$.

Page 106 (§ 40)

21. $\frac{1}{2}\sqrt{2}$.
23. $\frac{1}{2}\sqrt{6}$.
25. $\frac{1}{2}(\sin 6x + \sin 2x)$.
27. $\frac{1}{2}(\sin 10\theta - \sin 6\theta)$.
29. $-\frac{1}{2}(\cos 2B - \cos B)$.
31. $\frac{1}{2}(\sin 8x - \sin 4x)$.
33. $\frac{1}{2}(\cos 4x + \cos x)$.
35. $\frac{1}{2}(\sin 4x - \sin x)$.
37. $\frac{1}{2}(\cos 6B + \cos 2B)$.
39. $\frac{1}{2}(\sin 5\alpha - \sin \alpha)$.
41. $\frac{1}{4}$.
43. $\frac{1}{4}(2 - \sqrt{3})$.
45. $-\frac{1}{4}(2 + \sqrt{3})$.

Page 115 (§ 44)

3. $26°\,33'.9$, $63°\,26'.1$, $206°\,33'.9$, $243°\,26'.1$.
5. $30°$, $90°$, $150°$, $270°$.
7. $30°$, $150°$.
9. $22°\,30'.0$, $52°\,1'.1$, $112°\,30'.0$, $142°\,1'.1$, $202°\,30'.0$, $232°\,1'.0$, $292°\,30'.0$, $322°\,1'.1$.

Page 116 (§ 44)

11. $30°$, $150°$.
13. $26°\,48'.8$, $73°\,44'.0$, $106°\,16'.1$, $153°\,11'.3$, $206°\,48'.8$, $253°\,44'.0$, $286°\,16'.1$, $333°\,11'.3$.
15. $30°$, $90°$.
17. $143°\,7'.8$.
19. $166°\,42'.5$, $299°\,33'.1$.
21. $60°$.
23. $40°\,12'.5$, $252°\,24'.7$.
25. $102°\,21'.0$, $195°\,43'.4$.
27. $12°\,21'.0$, $105°\,43'.4$.
29. $18°$, $90°$, $162°$, $234°$, $306°$.
31. $60°$, $180°$, $300°$.
33. $22\frac{1}{2}°$, $112\frac{1}{2}°$, $135°$, $202\frac{1}{2}°$, $292\frac{1}{2}°$, $315°$.
35. $18°$, $54°$, $90°$, $126°$, $162°$ $198°$, $234°$, $270°$, $306°$ $342°$.

37. 0°, 30°, 90°, 150°, 180°, 270°, 360°.

39. 0°, 60°, 90°, 180°, 270°, 300°, 360°.

41. 0°, 30°, 60°, 120°, 150°, 180°, 210°, 240°, 300°, 330°, 360°.

43. 225°, 345°.

45. 37° 49′.3, 292° 10′.7.

47.

θ	30°	150°	270°
r	$\frac{1}{2}a$	$\frac{1}{2}a$	$-a$

49.

θ	26° 33′.9	90°	206° 33′.9	270°
r	$\frac{4}{5}\sqrt{5}$	0	$-\frac{4}{5}\sqrt{5}$	0

51.

θ	145° 10′.2	214° 49′.8
r	1.17915	1.17915

53.

y	15°	75°	105°	165°	195°	255°	285°	345°
x	$\pm\sqrt{2}$	$\pm\sqrt{2}$	$\pm i\sqrt{2}$	$\pm i\sqrt{2}$	$\pm\sqrt{2}$	$\pm\sqrt{2}$	$\pm i\sqrt{2}$	$\pm i\sqrt{2}$

55. $\frac{1}{3}$. **57.** $\frac{1}{3}$.

Page 117 (§ 44)

59. 45°. **61.** $\frac{1}{7}(4 \pm \sqrt{65})$. **63.** $\frac{1}{2}$ arc sin $\frac{36}{85}$.

Page 117 (General Exercises)

1. $-\frac{2184}{7225}$.

3. $-\frac{7}{25}$.

5. $-\frac{1}{5}\sqrt{5}$.

7. $\frac{18}{17}$.

9. (a) $-\frac{12}{49}\sqrt{5}$;
(b) $\frac{41}{49}$;
(c) $-\frac{12}{41}\sqrt{5}$;
(d) $-\frac{984}{2401}\sqrt{5}$.

Page 118 (General Exercises)

29. $m\sqrt{1-n^2} - n\sqrt{1-m^2}$. **31.** ctn $\frac{\alpha+\beta}{2}$. **33.** $\frac{h+k}{1-hk}$.

Page 119 (General Exercises)

35. $2\,ab\sqrt{1-a^2}+(2\,a^2-1)\sqrt{1-b^2}$.

37. x.

39. IV quadrant; $-\frac{4}{5}$.

41. No; 18° 26′.1, 90°, 198° 26′.1, 270°.

43. $\frac{1}{2}(n \pm \sqrt{n^2-8})$.

45. $0, \pm\frac{1}{3}\sqrt{3}$.

47. 1.

49. $2 \pm \sqrt{6}$.

51. 33° 41′.4, 213° 41′.4.

53. 22° 30′.0, 112° 30′.0, 202° 30′.0, 292° 30′.0.

55. 0°, 30°, 150°, 180°, 360°.

57. 9° 44′.0, 151° 20′.6.

59. $22\frac{1}{2}°$, 45°, $112\frac{1}{2}°$, $202\frac{1}{2}°$, 225°, $292\frac{1}{2}°$.

61. 150°.

63. 60°, 240°.

65. 45°, 135°, 225°, 315°.

67. 54°, 90°, 126°, 198° 270°, 342°.

69. 30°, 90°, 105°, 150° 165°, 210°, 270°, 285° 330°, 345°.

71. 158° 41′.7, 254° 26′.1.

Page 120 (General Exercises)

73. 30°, 90°, 150°, 210°, 270°, 330°.

75. 0°, 120°, 240°, 360°.

77. $4\sqrt{7}$ in.

CHAPTER V

Page 122 (§ 46)

5. (a) $K=\dfrac{a^2\sin B\sin C}{2\sin A}$; (b) $K=\dfrac{b^2\sin A\sin C}{2\sin B}$;

(c) $K=\dfrac{c^2\sin A\sin B}{2\sin C}$.

Page 128 (§ 47)

1. $C=62°\,38'.1$
$b=35.641$
$c=33.743$.

3. $A=64°\,4'.1$
$C=48°\,23'.6$
$a=23.861$
$K=218.72$.

5. No solution.

7. $C=35°\,56'.7$
$a=72.150$
$c=67.670$.

9. $B=67°\,33'.0$
$C=57°\,45'.2$
$c=8.8912$
$K=35.249$,
or
$B'=112°\,27'.0$
$C'=12°\,51'.2$
$c'=2.3386$
$K'=9.2713$.

11. $A=33°\,58'\,56$
$a=0.021497$
$c=0.028199$.

13. $B = 47° 14'.9$
$a = 124.23$
$b = 101.77.$
$A' = 16° 3'.5$
$C' = 122° 24'.9$
$a' = 0.18273.$

15. $A = 80° 53'.3$
$C = 57° 35'.1$
$a = 0.65227,$
or

17. $C = 138° 20'.7$
$a = 62.004$
$b = 28.993$
$K = 597.40.$

Page 129 (§ 47)

19. $B=65° 42'.7$
$b=14.738$
$c=15.946$
$K=65.411.$

21. $A=48° 26' 22''$
$B=18° 25' 5''$
$b=41.804.$

23. $K=920.90,$
or $K'=51.680.$

25. $K=0.00029948.$

27. $K=0.12058,$
or $K'=0.033780.$

29. $K=1902.9.$

Page 130 (§ 48)

3. $\cos A = \dfrac{b^2 + c^2 - a^2}{2\,bc}$; $\cos B = \dfrac{a^2 + c^2 - b^2}{2\,ac}$;
$\cos C = \dfrac{a^2 + b^2 - c^2}{2\,ab}.$

Page 131 (§ 49)

1. $c = 23.619.$

3. $B = 50° 7'.9.$

5. $b = 155.50.$

7. $A = 104° 28'.6$
$B = 28° 57'.3$
$C = 46° 34'.0.$

9. $A = 33° 12'.2$
or $33° 12'.3$
$B = 27° 9'.1$
or $27° 9'.0$
$c = 66.657.$

11. $A = 71° 4\ .5$
$B = 18° 55'.4$
$C = 90° 0'.0.$

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13. $A = 120° 0'.0$
$B = 21° 47'.2$
$C = 38° 12'.8.$

15. $K = 350,980.$

17. $K = 638.80.$

Page 135 (§ 51)

1. $A=77°\,28'\,29''$, $B=44°\,57'\,23''$, $c=238.75.$

3. $B=114°\,3'\,9''$, $C=54°\,13'\,25''$, $a=247.31$, $K=111,500.$

5. $A=43°\,0'.1$, $B=94°\,40'.1$, $c=110.48.$

7. $A=95°\,25'\,23''$, $C=19°\,47'\,23''$, $b=0.26904.$

9. $A=37°\,44'\,56''$, $C=44°\,4'\,16''$, $b=9.1648$ or 9.1645, $K=18.066.$

11. $B=18°\,40'.5$, $C=16°\,22'.3$, $a=3.9215.$

13. $K=0.00038330.$ **15.** $K=1736.2.$ **17.** $K=1.2086.$

Page 141 (§ 54)

1. $A=43°\,44'.4$, $B=32°\,49'.6$, $C=103°\,26'.0$, $K=1485.1.$

3. $A=59°\,2'.6$, $B=72°\,14'.8$, $C=48°\,42'.8$, $K=6201.1.$

5. $A=39°\,43'.0$, $B=98°\,40'.4$, $C=41°\,36'.6$, $K=146,690.$

7. $A=67°\,15'.0$, $B=61°\,7'.0$, $C=51°\,38'.0$, $K=5.0531.$

9. $A=54°\,18'.6$, $B=48°\,8'.2$, $C=77°\,33'.4$, $K=1,618,800.$

11. $K=146,690.$ **13.** $K=0.027887.$

Page 142 (§ 55)

1. 118.43 sq. in.; 26.795 sq. in.

3. 2.9615 sq. ft.; 3.2616 sq. ft.

5. 44.179 sq. in.; 1.1273 sq. in.

7. 2.9741 sq. ft.; 0.27757 sq. ft.

9. 1120.9 sq. cm.; 1364.5 sq. cm.

11. 507.00 sq. cm., 169.85 sq. cm.

13. 633.23 sq. in. **15.** $84°\,7'\,39''.$

Page 143 (§ 55)

17. 1026.3 gal.

Page 143 (General Exercises)

1. $B=38°\,35'\,8''$, $C=29°\,57'\,15''$, $c=3.7558.$

3. $A=45°\,47'.0$, $B=77°\,45'.4$, $C=56°\,27'.6$, $K=0.12126.$

5. $A=22°\,41'.6$, $B=47°\,53'.7$, $c=31.781.$

7. $A = 97° 31'.2$
$B = 33° 40'.6$
$C = 48° 48'.2$.

9. $A = 45° 12'.4$
$B = 77° 21'.9$
$C = 57° 25'.7$.

11. No solution.

13. $A = 60° 20'.8$.
$B = 51° 51'.8$
$C = 67° 47'.2$
$K = 3251.3$.

15. $B = 32° 14'.0$
$C = 48° 48'.6$
$a = 524.91$.

17. No solution.

19. $A = 48° 52'.9$
$B = 36° 44'.7$
$C = 94° 22'.4$
$K = 4.5767$.

21. $A = 25° 58'.3$
$a = 0.46965$
or 0.46966
$b = 0.99052$
or 0.99054
$K = 0.15409$.

23. $A = 112° 11' 44''$
$B = 29° 37' 52''$
$a = 0.074915$.

Page 144 (General Exercises)

25. 1436.9 ft.
27. 44.244 mi.
29. 172.41 ft.
31. 874.82 sq. ft.
33. 6.7972 mi.
35. 2.0774 sq. in.; 14.221 sq. in.

Page 145 (General Exercises)

37. 12,389 sq. cm.
39. 10.459 cm.; 25.355 cm.; 30.210 cm.; 107° 36'.1; 126.39 sq. cm.
41. 701.12 sq. in.
43. 120.07 ft.
45. 128.14.

Page 146 (General Exercises)

47. $83\frac{1}{3}$ ft.
49. 944.54 ft.
51. 703.07 ft.; 48° 33' 1''.
53. 2415.4 ft.

Page 147 (General Exercises)

55. 5318.0; 875.52.
57. 2553.6 ft.
59. 334.74 ft.
61. 2769.4 ft.

Page 148 (General Exercises)

65. N 40° 54'.5 E; 6 min. 15 sec.

CHAPTER VI

Page 162 (§ 59)

1. 0.51.
3. -1.79.
5. 0.24.
7. 1.71.
9. 0; 0.88.

Page 163 (General Exercises)

21. 1.28.
23. -1.66.
25. 1.79.
29. 1.03.
31. 2.11.
33. 1.61 in.
35. 0.86 ft.

Page 164 (General Exercises)

37. (a) 1.08; (b) 1.38.
39. 8.50 ft.
41. 42.7 in.; 108.7 in.

CHAPTER VII

Page 166 (§ 61)

1. (a) $\log_5 25 = 2$; (c) $\log_9 27 = \frac{3}{2}$; (e) $\log_{\frac{1}{8}} 0.25 = \frac{2}{3}$; (g) $\log_{27} 81 = \frac{4}{3}$; (i) $\log_{22} 1 = 0$.

2. (a) $6^3 = 216$; (c) $15^\circ = 1$; (e) $(\frac{1}{8})^{\frac{2}{3}} = \frac{1}{4}$; (g) $9^{1.5} = 27$; (i) $27^{\frac{4}{3}} = 81$.

3. $3, -\frac{1}{2}, \frac{3}{2}, 0, -\frac{5}{2}, 1$.

5. (a) $\frac{1}{2}$; (c) $\frac{2}{3}$; (e) $-\frac{3}{2}$; (g) $-\frac{2}{3}$; (i) -2.

6. (a) 4; (c) $\frac{1}{81}$; (e) 32; (g) $\frac{5}{2}$; (i) $\frac{1}{7}\sqrt{7}$.

Page 169 (§ 62)

5. $\frac{1}{2}\log_{10} 13 - 5\log_{10} 7 - \log_{10} 84$.

Page 170 (§ 62)

7. $\log_{12} \pi + 2\log_{12} r + \log_{12} h - \log_{12} 3$.

9. $2\log_e 28 + \frac{2}{3}\log_e 100 - \frac{1}{5}\log_e 219$.

11. $\log_{10} (\frac{1}{6}\pi d^3)$.

13. $\log_e \dfrac{x^{\frac{1}{2}} \cdot z^{1.2}}{y^{\frac{2}{3}}}$.

15. 1.62325.
17. -1.38021.
19. 0.05115.
21. 0.29542.
23. 4.65770.
25. $y=10^{x^2}$.
27. $y=a^{-\frac{1}{x}}$.
29. $y=\sqrt{x+1}\cdot a^{-x^2}$.
31. 2.
33. $\frac{52}{15}$.
35. $\frac{39}{25}$.
37. 0.

Page 173 (§ 64)

1. (a) 1; (c) 0; (e) -4 or $6-10$; (g) 2; (i) 1; (k) 1.

Page 177 (§ 65)

11. $9.88012-10$.
13. 2.99541.
15. $9.63062-10$.
17. 1.46388.
19. $8.21582-10$.

Page 179 (§ 65)

9. 4.8356.
11. 593.99.
13. 72.457.
15. 230,590.
17. 10,003.

Page 186 (§ 67)

1. 0.23337.
3. 26.734.
5. -0.11941.
7. -24.158.
9. 385.98.
11. 0.000018496.
13. 1.5693.
15. -0.35036.
17. 0.96614.
19. 0.024917.
21. -0.89318.
23. 0.29072.
25. 28.420.
27. 4.6784.
29. -0.12135.
31. 0.25378.
33. -0.23442.

Page 187 (§ 67)

35. 11.604.
37. 2.8624.
39. 1.6639.
41. 0.00065177.
43. -0.11428.
45. 0.77386.
47. 15.290.
49. 1.3573.
51. 0.011884.
53. (a) 1165.3; (b) 1188.4.

Page 188 (§ 67)

55. 1.0104.
57. $2470.60.